Moduli spaces of square-tiling deformations and single-tile real line tilings

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Garrett Proffitt
mentored by
Dr. Matthias Weber

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Abstract

Classifying Single-Tile Periodic Tilings of the Real Line and Realizing the Deformation Spaces of Two and Three Square Periodic Tilings of the Plane through Combinatorial Structure.

1 Introduction

A tiling can be thought of as an assortment of objects that covers a space cleanly and without any gaps. A periodic tiling can be thought of as a base object that is a subset of the space (what we will call a fundamental domain) being translated around the space by a consistent method of instructions (what we will deem translation by the group of translational vectors). Generally, in $\mathbb{R}^n$, we can think of a fundamental domain as an n-dimensional object (or subset of $\mathbb{R}^n$) with a set of n translational vectors. We will be looking at periodic tilings in $\mathbb{R}$ and $\mathbb{R}^2$, so we will only need one or two translational vectors.

Figure 1: Example of a periodic tiling in $\mathbb{R}$

More specifically, in $\mathbb{R}^2$, we will be looking at periodic tilings by squares, most notably tilings by one, two, and three squares.

As for periodic tilings of $\mathbb{R}$, we will be looking at tiles consisting of disjoint intervals. Many examples will be given for both. (Note we cannot show all of $\mathbb{R}$ or $\mathbb{R}^2$ in our tiling, so we will only show a subset of these regions being tiled.)

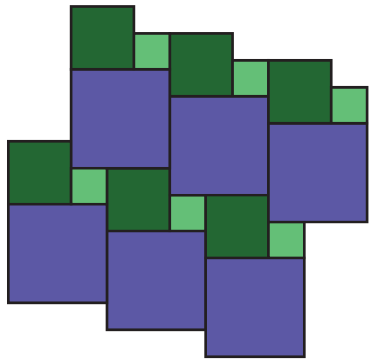

Figure 2: Example of a tiling in $\mathbb{R}^2$

2 Tilings of the Real Line

2.1 Defining the Tiling

Given any length interval in $\mathbb{R}$, call length L, we can subdivide the interval into n parts, say, of lengths $a_1, ..., a_n$ such that the sum of the a_i's equals L. Note the a_i's do not have to be in any specific order.

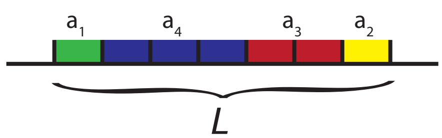

Figure 3: A divided interval of length L

We can then deform the intervals in a modularly arithmetic specific manner (breaking the parts up and putting them apart from each other, called a **lift** mapping) to create a union of disjoint intervals. We call this an n-**prong tile**.

Definition 2.1. Let $a_1, ..., a_n$ represent interval lengths in $\mathbb{R}$. An n**-prong tile** is a union of disjoint intervals $a_1, ..., a_n$ (a_i's in increasing order) with $a_1 + ... + a_n = L$ such that the tile modulo L is a single connected tile of length L.

Figure 4: Creating an n-prong tile ($n = 4$) from a single interval of length L

Let the a_1 length interval always be the first disjoint interval (we know we can do this by simply moving any interval in front of a_1 to the back (e.g. $a_5a_{10}a_1a_n...a_{n-25}$ is equivalent to $a_1a_n...a_{n-25}a_5a_{10}$ by modular arithmetic mod

L). Then this n-prong tile will tile $\mathbb{R}$ by translation with period L. We would like to say this is the only way to tile $\mathbb{R}$ with translates of one tile, but it will not be proven in this paper (see the results of Lagarias and Wang in [1] for showing how all tilings of the real line by a single tile will eventually be periodic).

2.2 Examining the Tiling

Note that with n disjoint intervals, (we will refer to them as prongs, numbered in increasing order: "1, 2, . . . , n"), we have one prong that is fixed (namely prong 1), which we must have in order to define our n-prong tile.

Figure 5: Because of the numbering of the prongs (1-2-3, 2-3-1, and 3-1-2), the above three tiles are equivalent; they are just starting at different prongs. We fix prong 1 as the starting prong to make sure we don't repeat the same tilings.

This means the other $n-1$ prongs can be patterned in any desired order over the period-length interval (i.e. when the tile is taken modulo L), which allows for $(n-1)!$ **classes** of countably infinite many different tilings. We control the different tilings by modulating the gap sizes, or if the **prong pattern** is in consecutive order (e.g. 1, 2), we can set the gap between the two equal to zero and make an $n-1$ prong tile, or set the gap equal to L, and by modulating this length we get $0+L+L+...+L$ (m times), or simply mL. (For modulating other gaps, we take the shortest length between the consecutive prongs (say $a_1 + a_3$ or something) then go around the interval again, which will of course add on L to the shortest gap length, and we can do this as many times as we please.)

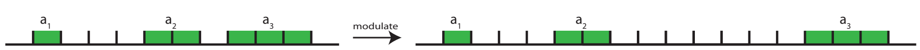

Figure 6: Modulating the second gap in a 3-prong tile

Also it should be noted that gaps can be modulated independently. If we call gaps b and call the gap after each a_i, (for $i = 1, ..., n$) b_i (so that b_n is the gap after the end prong a_n), we can associate an m_i in $\mathbb{N}$ to each b_i to indicate how modulated a gap is, and m_i does not necessarily have to equal m_j. This ensures there are countably infinite different tilings for each class of tiles.

Figure 7: We use the same tile as in the previous example. This time, the first gap is modulated once and the second gap is modulated twice. We also show it tiles.

Here are the definitions of some of the terms emphasized above:

Definition 2.2. The **prong pattern** of an n-prong tiling is the numerical ordering of the prongs when the tile is taken modulo L. The prong pattern determines which tiling **class** we are in, which is useful in determining if two tilings are **equivalent**.

Definition 2.3. Two tilings are in the same **class** if their prong patterns are the same. If two tilings are in the same class, they are said to be **equivalent**.

2.3 Building up the Classes

Let us look at the classes for each n-prong tile, starting with $n = 1$:

For a 1-prong tile, all we do to tile is shift by the period (or interval) length L. We can call this **Class 0** or "class-less" since the prong pattern is always "1, 1, 1, 1, . . .", and no other tilings have the same prong number adjacent to one another.

Figure 8: Example of a Class 0 tiling

For a 2-prong tile, we have prong number 1 fixed in the first spot, so prong number 2 must come next, yielding $1! = 1$ class(es), so there is only one class of countably infinite tilings. The prong pattern will always be "1, 2, 1, 2, . . .". Let us examine further:

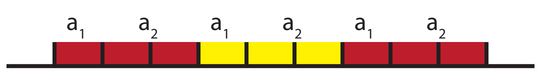

Figure 9: A two prong tile with a gap length of zero

We can have gap $b_1 = 0$, but that reduces to a class-less 1-prong tile. (Note the obvious fact that combining any two prongs by setting $b_i = 0$ will reduce an n-prong tile to an $(n-1)$-prong tile, drastically reducing the number of classes for decent-sized n (say $n \geq 4$).) To make things interesting, we take $b_1 = a_2 + a_1$ and get:

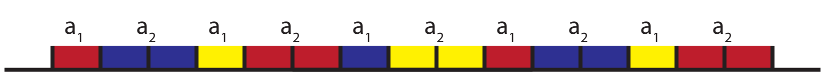

Figure 10: A two prong tile, tiling by translation by $a_1 + a_2$

As previously stated, we can take $b_1 = m_1(a_2 + a_1)$ to get the countably infinite many tilings in this class. Note the increasing modular pattern "1, 2, 1, 2, . . ." where the prong number is increasing until all the prongs are named (e.g. “1, 2, 3, 1, 2, 3, . . .” for 3-prongs and “1, 2, 3, 4, 1, 2, 3, 4, . . .” for 4-prongs). This pattern shows up as a class for all n-prong tilings, $n \geq 2$, so we call this **Class 1**.

Definition 2.4. For any n-prong tile, $n \geq 2$, a tiling with prong pattern “1, 2, 3, 4, . . . , n” is defined to be a **Class 1** tiling.

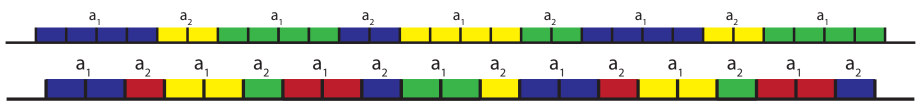

Figure 11: Examples of 2-prong Class 1 tilings; note the second example has its gap modulated

For a 3-prong tile, we have prong number 1 fixed in the first spot, leaving prongs 2 and 3 free, yielding $2! = 1$ classes with prong patterns “1, 2, 3, 1, 2, 3, . . .” or “1, 3, 2, 1, 3, 2, . . .”. We have already looked at the first and named it Class 1 (and it is important to note Class 1 tilings have the longest unmodulized gaps, for all the gaps are always equal to at least L). We now must look at this second interesting prong pattern that is decreasing modularly (like “3, 2, 1, 3, 2, 1, . . .” with 1 moved to the front by convention):

Figure 12: Showing the lift mapping for a Class 2, 3-prong tile

Completely opposite of the Class 1 tilings, these **Class 2** tilings have the shortest unmodulized gaps (in the case of the 3-prong tiles). As with Class 1, Class 2 tilings appear for every n-prong tiling (for $n \geq 3$ this time).

Definition 2.5. For any n-prong tile, $n \geq 3$, a tiling with prong pattern “1, n, $n-1$, $n-2$, . . . , 3, 2” is defined to be a **Class 2** tiling.

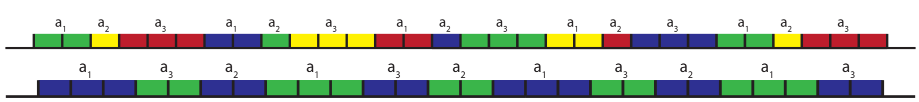

Figure 13: Examples of 3-prong tilings; the first is a Class 1 tiling and the second is a Class 2 tiling

For a 4-prong tile, we have prong number 1 fixed in the first spot, leaving three free prongs, yielding $3! = 6$ classes, and this is where the classes start to become overwhelming. We only name Class 1 and Class 2 and call the other classes by the prong pattern they have. For the 4-prong tiles, the six classes are:

Class 1 (1,2,3,4)
Class 2 (1,4,3,2)
Class 1-2-4-3
Class 1-3-2-4
Class 1-3-4-2
Class 1-4-2-3

As always, the gaps can be modulized, and the period L is the sum of the prong lengths.

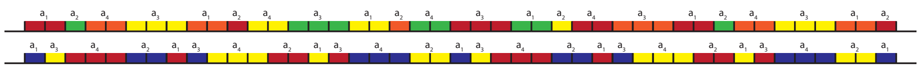

Figure 14: Examples of 4-prong tilings; the first is a Class 1-2-4-3 tiling and the second is a Class 1-3-4-2 tiling

We can keep going, to five, six, and seven prong tiles, but, in general, for n-prong tiles, we fix the first prong, leaving $n-1$ free prongs, which yield $(n-1)!$ classes. Class 1 is always included for $n \geq 2$, and Class 2 is always included for $n \geq 3$. Gaps can always be modulized independently, and the tiles are always translated by L, the sum of the prong lengths.

Figure 15: Examples of n-prong tilings; the first is a Class 1-3-5-2-4, 5-prong tiling, and the second is a Class 2, 6-prong tiling

2.4 Number Classification by Tilings Theorem

With the above information we have gathered empirically, we can now form a theorem relating these tilings of the real line to number theory, namely, that we can produce countably infinite real line tilings (ignoring the obvious uncountably infinitely many choices for each prong's length) for any finite ordered list of natural numbers. For ease, we will require the list of numbers to always start with 1 to avoid any repetition.

Theorem 2.6. *Given a finite ordered list N of n natural numbers beginning with 1, there exist countably-infinite many n-prong tiles (ignoring the choice of prong length) whose prong patterns are the same as the ordered list N.*

Proof. The proof simply follows from what we have already found. If we have our ordered list N that begins with 1, we can relate this list to a prong pattern of some n-prong tile. To each number i in N, we can associate a real value a_i in order to construct this tile. As before, let $L = a_1 + ... + a_n$ and then create an L-length interval broken up into these n parts with a prong pattern equivalent to the list N. Then we can lift the tile from its modulo L state into a tile with n prongs such that we tile $\mathbb{R}$ periodically. The "countably-infinite many" part comes from the fact that we can modulate the size of each gap to whatever we please. $\square$

This theorem is useful for giving us a geometric representation of a finite pattern of natural numbers, and it is interesting to note that for *any* ordered list of natural numbers, we can find an n-prong tiling (and not just one, but *always* countably infinitely many).

3 Tilings by Squares

3.1 Definitions and Examples

Recall in the introduction we mentioned the criteria for a tiling. For tilings by squares, the space we are looking to tile is $\mathbb{R}^2$ and our **fundamental domain** is a square or a union of several squares (without any gaps). The **translational vectors** are two linearly independent vectors in $\mathbb{R}^2$, inherently with an x component and a y component. Below is an example of a tiling by one square:

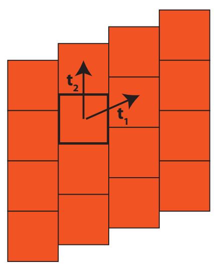

Figure 16: A tiling by one square with translational vectors $\vec{t_1}$ and $\vec{t_2}$

Note the fundamental domain in this case is just one square, but the situation changes with two squares:

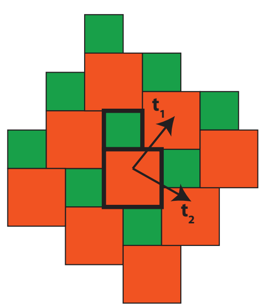

Figure 17: A tiling by two squares with translational vectors $\vec{t_1}$ and $\vec{t_2}$

It would be wise to define what we mean by fundamental domain and translational vectors in this case:

Definition 3.1. Given a topological space X and a group G acting on it, we can find that the images of a single point under the group action form an orbit, and a **fundamental domain** is a subset of X that contains exactly one point from each orbit.

This definition can be simplified for our purpose of periodically tiling $\mathbb{R}^2$ by squares since we are looking at $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by translation, which will always form a lattice.

Definition 3.2. For $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by translation (the case for our square tilings), a **fundamental domain** of a tiling is the most irreducible and irredundant piece of the tiling such that the whole tiling can be determined (in accordance with translations by a constant set of two translational vectors). We define irreducible to mean that no subsets of the fundamental domain will be able to tile the plane by itself, and we define irredundant to mean the intersection of any two fundamental domains is the empty set. In our case with a tiling by n squares, this means each of the n squares can only be represented once in the fundamental domain.

Definition 3.3. In general, for a vector space V of dimension n, the set of **translational vectors** is the consistent set of instructions that determines the n-linearly independent directions in which the fundamental domain will move, and this set of instructions is repeated indefinitely until the entire space is tiled. For $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by translation, the set of translational vectors are two linearly independent vectors $\vec{t_1}$ and $\vec{t_2}$ in $\mathbb{R}^2$.

Looking back at our first two-square example, this becomes obvious:

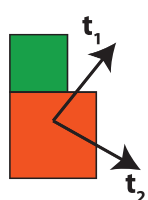

Figure 18: A single tile of two squares that tiles $\mathbb{R}^2$

The "L"-shaped fundamental domain is being translated by the vectors $\vec{t_1}$ and $\vec{t_2}$ throughout all of $\mathbb{R}^2$, and we can see this will tile the plane periodically. We can prove this by looking at the fundamental domain and seeing where another copy of it would have to go to tile periodically. In this case, we look at the corner where the "L"-shape bends and look at what can fill that corner. The only possible fit is the bottom left corner of the fundamental domain, so we attach another copy there and continue:

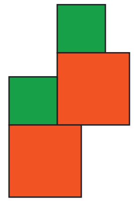

Figure 19: The two tiles must fit together in this manner

Note that this automatically determines one of the vectors, since we must do this for all copies and so we will eventually form an infinite band.

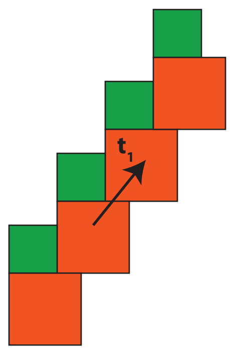

Figure 20: One translational vector is determined

To find the other vector, we must note how the long bands can fit together. Since there is one band, there must be another. We look at the zig-zagging pattern on the bands, and note how the two bands fit together.

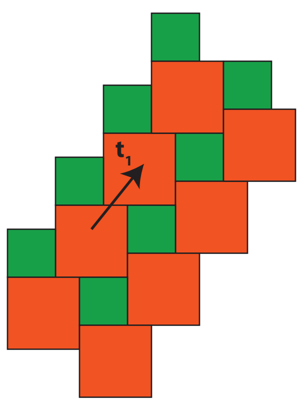

Figure 21: Two bands can only fit together in this manner

Since these two bands fit together, we can connect all bands together and get a periodic tiling of the entire plane.

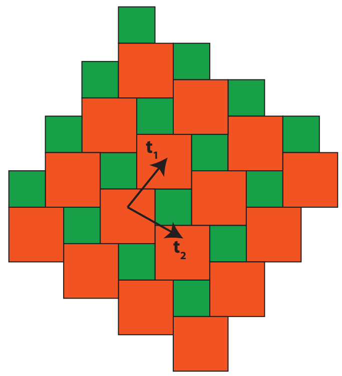

Figure 22: The second vector is determined, and we tile $\mathbb{R}^2$

We can construct a useful proposition of the process we just went through for all "L"-shaped tiles:

Definition 3.4. A fundamental domain (or tile) is "L"-shaped if it consists of six vertices, five of which form outward-oriented corners and one of which forms an inward-oriented corner (as one would expected for any block "L"). A **perfect** "L"-shaped fundamental domain is one that can take no other forms when tiled, i.e. its tiling cannot have an alternate fundamental domain that is a rectangle or a "Z"-shape (we will not discuss "Z"-shaped tiles however).

Proposition 3.5. *Given a perfect "L"-shaped fundamental domain in* $\mathbb{R}^2$*, there exists only one unique tiling of* $\mathbb{R}^2$

Proof. To tile, all gaps must be filled. That means the one inward corner of our "L" shape must be filled. However, the only possible translate that will fill that corner is diagonal, so that the corner opposite the inward one meets it cleanly (a figure above represents this step). If we don't do this, we will have a gap or an overlap near the inward corner. We can translate in this direction infinitely long to get an infinite band, and now we must find how the band interlocks.

Since we plan on tiling the entire plane, we must have another one of these infinitely long bands, and the two must cleanly match-up (otherwise we will have gaps or overlaps). Note the zig-zag patterns on either side of the bands. We can cleanly match-up the bands in multiple ways (just by shifting the bands against each other), but each of these match-ups produces the same tiling (see one of the above figures for a visualization). Continuing to add these bands on in this manner will tile the plane.

We tried all possibilities to create this tiling (looking at options for the inward corner and the zig-zag), but we can see there is only one tiling in which the "L"-shaped tiles will fit together. □

The process discussed above can be used to determined if a tile will actually tile the plane if it is not “L”-shaped, but as one can see the process can be quite lengthy and fairly inefficient. Because of this we introduce the new idea a **deformation space**.

Definition 3.6. Given a periodic tiling of the plane by squares whose sides are parallel to the axes, a **deformation space** is the set of all possible periodic tilings that can be obtained by a single translation or a single scaling of a subset of squares in the fundamental domain. Both the scaling and shifting are done in a continuous manner.

For ease, we will disregard any deformation that uniformly scales all parts of the fundamental domain, and we will disregard any rotations that are used to deform the tiling. Since this can be done with any tiling, it is best to mention it once but then ignore it due to triviality.

We can see what this looks like in our “L”-shaped example. We can deform this tile by scaling the smaller square on top until the two squares are of equal size.

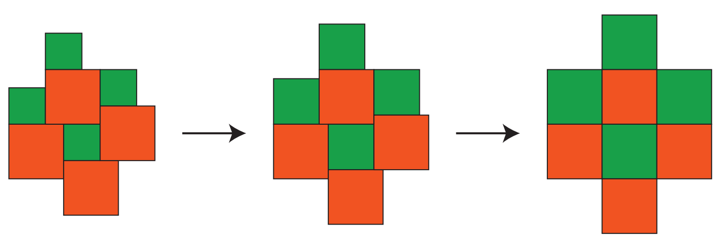

Figure 23: Deforming the “L”-shaped two square tiling by growing the small square

Note we can continue this deformation by now shrinking the square to the other side:

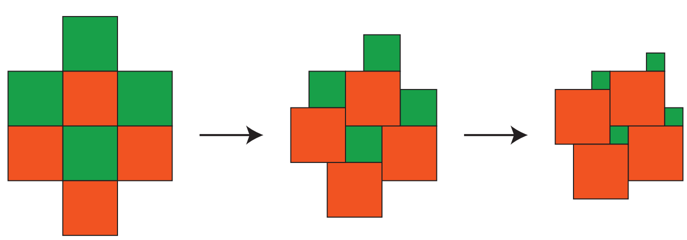

Figure 24: Deforming the equal-size two square tiling by shrinking the green square to the right side

If we continue this deformation, we will end up shrinking one square to a point, leaving us with one square. We can also start with our original tile and shrink the top square to a point, again leaving us with one square, the same one as in the first case. This completes a deformation space with two squares of different sizes. Note the importance of color in the example as well; a tiling by a single orange square and a tiling by a single green square are not the same.

To make sure a fundamental domain tiles, we still must go through the steps outlined above. Allowing for the deformation spaces gives us a better intuition on what may tile and what may not, and we are better able to understand the underlying structure of a **tiling system** with their help.

Definition 3.7. A **tiling system** for a periodic tiling by n squares is the complete classification of all possible tilings by n squares. That is, it is the union of all existing deformation spaces.

The tiling system is usually made evident by drawing the entire moduli space, which we will now define.

3.2 Moduli Spaces

After the insight deformation spaces give us about the underlying structure of a tiling system, one begs the question, "are there multiple deformation spaces?" and "how are they related?". We introduce the concept of **moduli space** to answer these questions and give us a full view of the structure of the tiling system.

Definition 3.8. A **moduli space** is a geometric object whose points uniquely represent another mathematical object (in this case different periodic tilings by squares) that are connected by isomorphism classes (in this case they are of the same deformation space). Furthermore, note different moduli spaces can intersect in different ways.

In our case, we use moduli spaces to represent deformation spaces, where each point in the moduli space represents a unique tiling. It is important to note that two different points cannot represent the same tiling. A given deformation space maps bijectively to a corresponding moduli space; if two deformation spaces share the same tiling, these two deformation spaces must intersect at a point.

In the end of the previous section, we mapped out the deformation space of the "L"-shaped two-square tile. We can represent this as a moduli space in the form of a loop. Since the translational vectors are uniquely determined by the shape of the fundamental domain (i.e. it only admits one specific tiling), we only have a one-parameter family (save for scaling and trivial rotation) since we can only choose the size of the second square, hence the one-dimensional moduli space (the loop).

It follows that since all our deformation spaces will be "cyclic" in the one-, two-, and three-square cases, all our moduli spaces must be "cyclic", or closed into a loop for each dimension the deformation space admits. (Here, "cyclic" means that continuing the deformation will return us to the tiling with which we started, same as what we did in the previous section. There is a slight exception in the case where we reduce our tiling to a fewer number of squares, and we will address that issue in this section.)

For dimension greater than or equal to two, it would seem that a sphere may fit this description. However, in a sphere the "loop size" will vary depending on where you are on the sphere. If we look at all loops in the x-direction in say the two-sphere, we notice the loop is full at the equator but shrinks to a point as we increase or decrease our value for y. This cannot be; it doesn't follow the structure of the deformation space we want. A torus, however, does keep a constant-sized loop for each dimension, indicating that continuing a deformation will eventually return us to the tiling with which we started. (Note this also makes more sense topologically. The loop in the one-dimensional case is simply S^1, and for an n-dimensional moduli space, we want all n-dimensions to have the independent property of following a loop. We can then take the Cartesian product of all these S^1 objects to maintain the "independence" of each dimension, giving us $S^1 \times S^1 \times ... \times S^1$, which is homeomorphic to the n-torus.) Therefore in our cases of one-, two-, and three-square tilings, the moduli spaces will either be loops or tori, and we will see that the interest lies in how they are all connected.

3.2.1 Moduli Space for One Square

Let us first look at the one-square moduli space. In determining how to begin looking at the moduli space, it is often best to start with the most basic tiling one can think of. In this case, we want to look at the "grid" tiling where all the squares line up (in the next section we will understand combinatorially why this is the case).

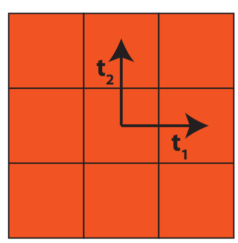

Figure 25: The basic "grid" tiling of one square

Notice that the fundamental domain in the one-square case is simply one square, so any scaling done to part of the fundamental domain will inherently tile the entire fundamental domain. Therefore we can eliminate the "independent scaling" deforming tactic mentioned previously and just look at shifts/translates. Notice that two shifts can occur by moving vertical bands up or down or moving horizontal bands left or right. Let us first look at moving the vertical bands and look at how the translational vectors are affected:

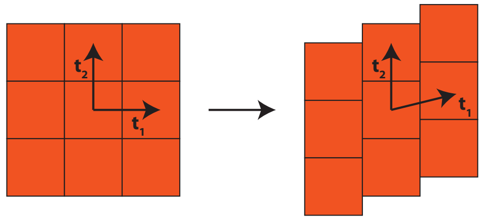

Figure 26: Deforming the one square "grid" tiling by a vertical shift

Right away note that the vertical vector stays the same, but the previously horizontal vector now has a y-component. Also note that there are no free parameters for this tiling. The new tiling created by the vertical shift still has two well-defined translational vectors coming from each fundamental domain (i.e. each square), so the tiling is determined uniquely.

We can continue this deformation until we reach the point where the tiling becomes a grid again, but note how the translational vectors have changed:

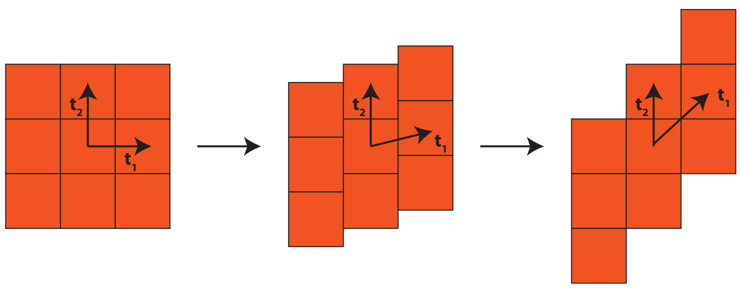

Figure 27: Continuing the vertical shift deformation

Fortunately, the first grid and the last grid are isomorphic by the group action by $SL_2(\mathbb{Z})$, so we have

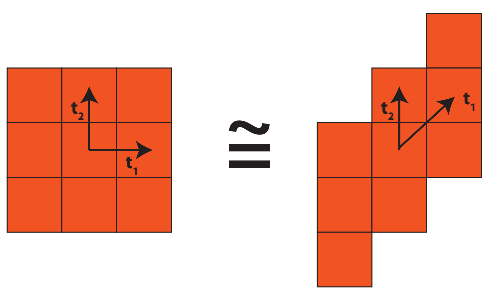

Figure 28: Isomorphic by group action by $SL_2(\mathbb{Z})$

and this shows that we have completed the loop for this moduli space.

Similarly, we can follow the same process for the horizontal band shift. There will be no free parameters, so we will again have a one-parameter family (another loop). Continuing the deformation, we will reach the original grid as before with different translational vectors, but again these tilings are isomorphic by the group action by $SL_2(\mathbb{Z})$. That closes the loop for the horizontal shift moduli space, and since there are no more deformations we can do (save for the trivial ones of scaling and rotating), we have a complete picture of all tilings by one square, and it turns out to be a nice figure eight:

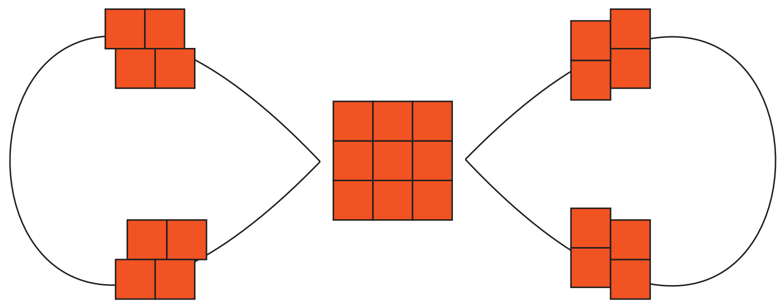

Figure 29: The entire one-square moduli space

3.2.2 Moduli Space for Two Squares

Things get more interesting with two-square tilings. As before, we want to start with the most basic tiling we can think of, which would be where all the squares are of the same size and line up in a grid. There are two colors this time instead of one, so we must think what the most basic and homogeneous way is to arrange the tiles. If we have long horizontal or vertical bands of the same color, we eliminate the homogeneous aspect we were going for, so we will instead arrange the initial tiling as a checkerboard pattern.

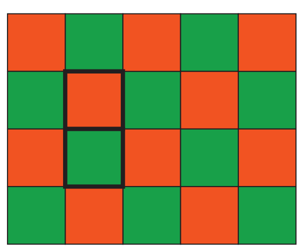

Figure 30: The checkerboard pattern two square tiling

Note there are four fundamental domains we can take from this tiling: a tall rectangle and a wide rectangle, both have a case where orange is the top-most or left-most color, and both have a case where green is the top-most or left-most color.

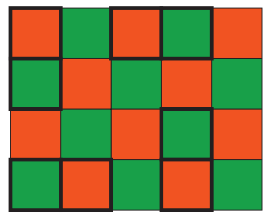

Figure 31: The checkerboard pattern with four possible fundamental domains highlighted

In this case (as we will see), the same tilings and deformation spaces are created with the four different fundamental domains, but we cannot always assume this, so it is important to make note of which fundamental domains we have available.

Let us first take the case where we scale one square to be smaller, yielding an "L"-shape tiling that we discussed in the previous section. We will deform our checkerboard tiling by shrinking the orange square relative to the green square. Note that we can shrink it two ways:

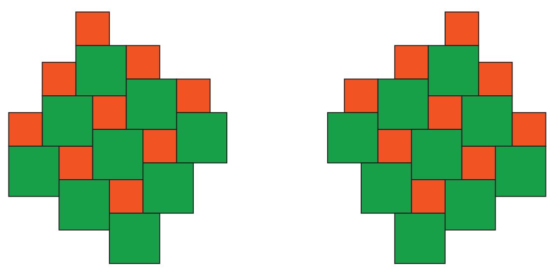

Figure 32: Both orientations of the shrinking orange square deformation

Also make note of the various fundamental domains for each tiling. Each of the tilings admits four fundamental domains, highlighted below:

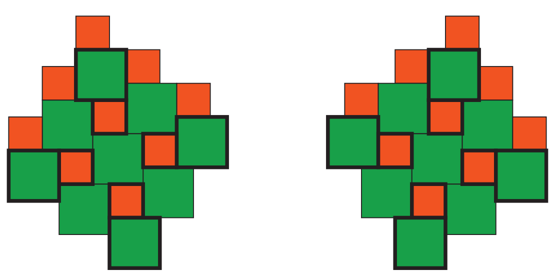

Figure 33: Both orientations of this deformation with fundamental domains highlighted

We can continue shrinking the orange square in both tilings until it becomes a point, giving us two green squares with an orange point on a corner (the corner on which the point lies depends on which fundamental domain we choose).

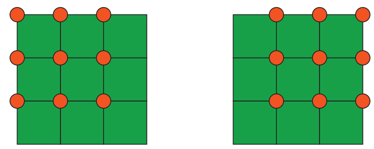

Figure 34: Both orientations in which the orange square shrinks to a point

However, it is obvious that these two new tilings are the exact same. If we look at the tiling that the green square and the orange point admits, we find that we have a grid of green squares with an orange point at every single vertex.

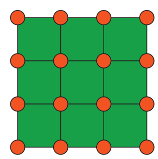

Figure 35: The single tiling at which the orange square has shrunk to a point

It now becomes unclear which green square/orange point orientation we started with, so we know that all orientations admit the same tiling. This holds true for every square with an accompanying point. Therefore we can essentially disregard the point, and this leaves us with a tiling by one square, whose "figure eight" moduli space we already know about. Then we have a complete loop with what we will call a **limit point** where we attach the moduli space for one square.

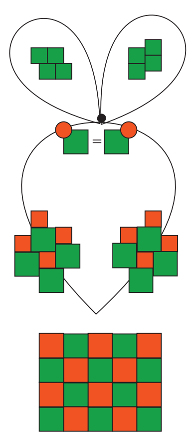

Figure 36: The moduli space in which we shrink the orange square

Definition 3.9. A **limit point** is a point in a moduli space for periodic square tilings by n squares ($n \leq 3$, in our case) where one or more of the squares is deformed into a point. At this point, we can attach the moduli space for square tilings by m squares (where m is the number of remaining squares that haven't been deformed to points). We perform this process to understand how all the different moduli spaces are connected.

We can repeat what we just did by shrinking the green square relative to the orange square, getting two different tilings depending on which side the green square was shrunk, and then getting a tiling by a single orange square where the green square is deformed to a point at the limit point in the loop. We then attach the moduli space for one square as stated in the above definition.

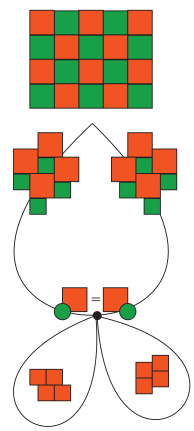

Figure 37: The moduli space in which we shrink the green square

Now that we have done all the independent scaling we could do, we must look for the possible shifts that can be done. Notice right away that a vertical band shift and a horizontal band shift is possible as in the one-square case. Let us first look at the horizontal band shift:

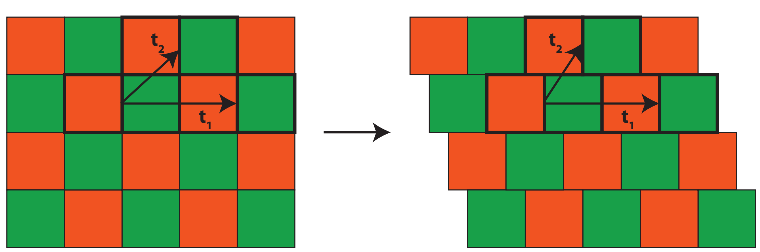

Figure 38: The horizontal shift deformation from the original checkerboard tiling

Note that as with all the previous deformation cases, the translational vectors are determined by the tiling. We have one horizontal vector and one diagonal vector. However, let us continue this horizontal shift. Eventually notice that the green and orange squares will be aligned in vertical bands. At this point, we could continue with our horizontal shift, eventually returning back to the checkerboard tiling and staying in the same moduli space, or we could vertically shift the same-color bands, entering a whole new moduli space. Since we already have a firm grasp on the first case, we will enter the new moduli space.

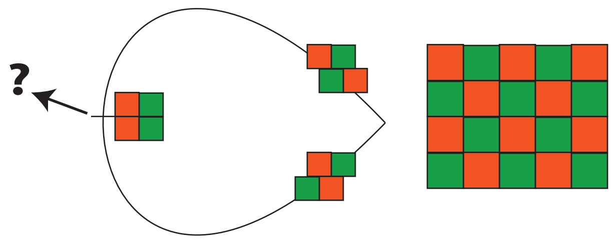

Figure 39: The moduli space for the horizontal shift deformation

The new way to deform this band tiling is to shift the vertical bands up or down. Let us do that and look at the translational vectors.

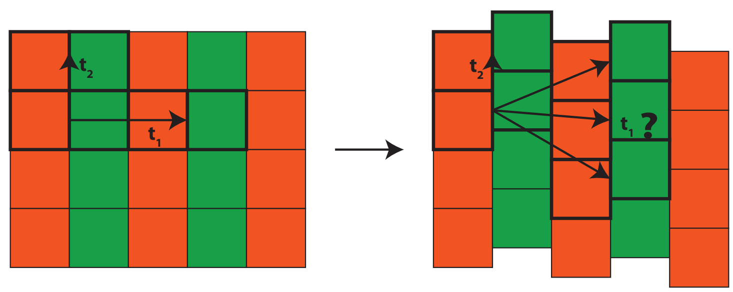

Figure 40: Shifting the vertical bands; note the one free vector

One of the vectors is defined, but the other can go anywhere past the adjacent green band. This gives us our first free translational vector in our moduli spaces, and since we already can shift the green band freely, we have two variables in this deformation space. This means instead of a loop we get a two-dimensional torus as a moduli space (the reasoning for this was explained earlier in the section).

We can repeat this whole process from the beginning at the checkerboard tiling, this time starting with a vertical shift deformation rather than a horizontal shift, and we will end up with horizontal bands of the same color, which when shifted against each other will also admit a two-dimensional torus as a moduli space.

These are all the deformations we can do with a two-square tiling, so we can combine all these moduli spaces into a single picture, giving us the entire system of periodic tilings by two squares.

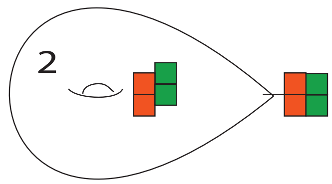

Figure 41: The toroidal moduli space for shifting the vertical bands

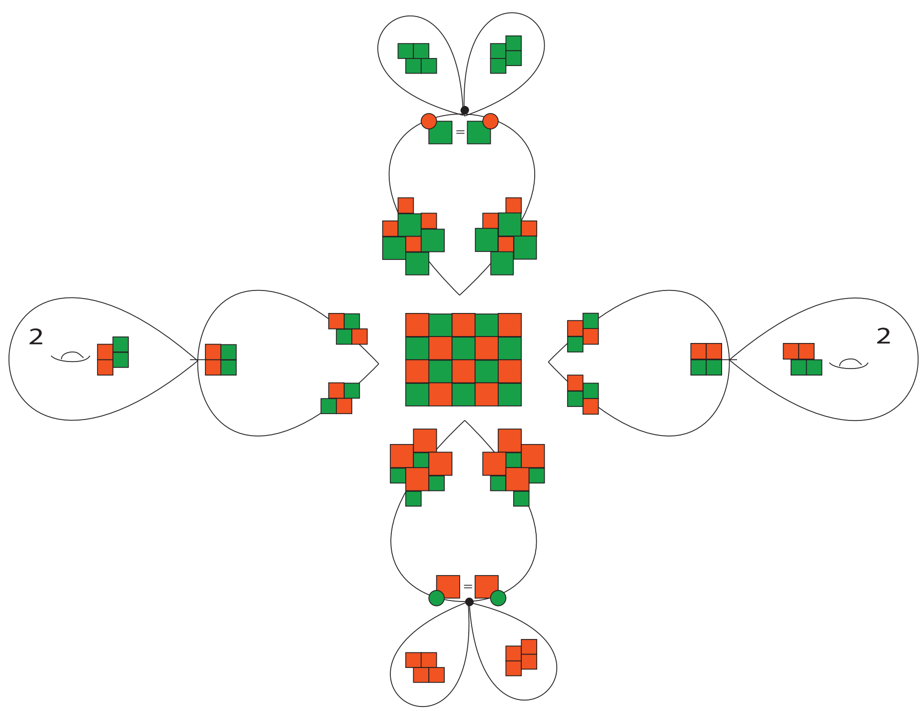

Figure 42: The entire two-square moduli space

It is important to notice that we can deform any tiling by two squares into another tiling by two squares without ever touching a limit point (the same applies for tilings by one square). This means the entire space of tilings by two squares is connected, which is a critical observation. Also, at the beginning of the discussion about the two-square space, we mentioned that the fundamental domain we look at can be any one of the four possible ones on the base tilings, and that any deformations we produce from the four will produce equivalent tilings. Instead of explaining with words, look now at the four pictures of the moduli spaces where we observe a different starting fundamental domain:

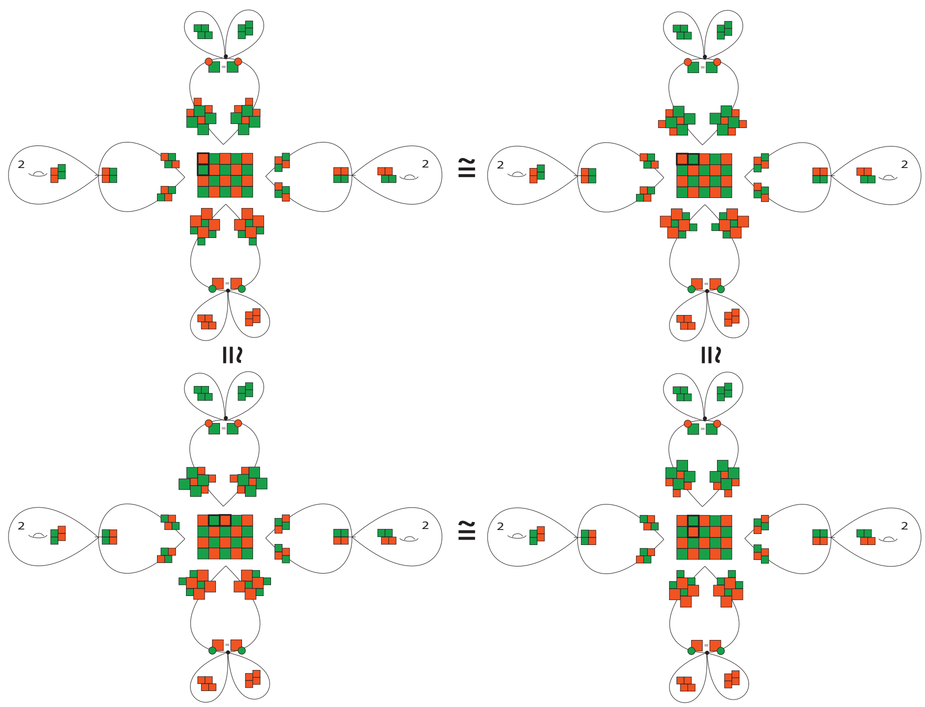

Figure 43: The four equivalent variations of the two-square moduli space, implying the choice of a fundamental domain does not matter for two squares

Note that for each deformation space, the tilings are the same, though different fundamental domains exist. The translational vectors for each case are different, but since the tilings are equivalent it does not matter. (Since each point in the moduli space uniquely represents a tiling, it does not matter if the translational vectors are different; it only matters that we have the same tiling.) There is always an “L”-shape with one color bigger than the other, there is always a checkerboard base tiling, there are limits to one square of each color, and there are the bands of color; each has multiple fundamental domains, but the tilings are equivalent. This brings us to an important proposition for two-square tilings:

Proposition 3.10. *Given a periodic tiling of the plane by two squares of colors a and b, we can take any fundamental domain we like and deform however we like and the deformation space (and thereby the moduli space) will be the same. That is, the choice of a fundamental domain does not affect the moduli space for the two-square tilings.*

Proof. The four illustrations above clearly depict the equivalence of the moduli (and deformation) spaces. Since the above moduli space represents all possible periodic tilings by two squares, the proposition holds true. □

This proposition becomes useful when studying the three-square tiling, since there are multiple places where a tiling at a limit point will have a different fundamental domain, but will produce the same tiling. Since the tilings are what we care about, we merely make these the same point to adhere to the moduli space's definition of unique representation.

3.2.3 Moduli Space for Three Squares

With tilings by three squares, everything breaks loose. We no longer have a simple, homogeneous, basic tiling with which we can start. Instead, we have four. As with the previous two cases, we find our beginning tilings by taking the most basic and homogeneous tilings, i.e. having all squares with the same size and having no two squares of the same color share an edge. Furthermore, it helps if we fix one of the square colors as the "lead color" in order to eliminate any redundant reordering of fundamental domains (e.g. an "orange-green-blue" horizontal rectangle as a fundamental domain will clearly admit the same tiling as a "green-blue-orange" and a "blue-orange-green" horizontal rectangle). However, reorderings are not the same as permutations (e.g. an "orange-green-blue" horizontal rectangle as a fundamental domain is *not* the same as an "orange-blue-green"). This is an important proposition.

Proposition 3.11. *Consider a periodic tiling by two or three squares in which the squares are all the same size and the fundamental domain can be represented as a rectangle that is the union of two or three squares (i.e. two or three squares tile a rectangle). The tilings will be equivalent if the colors in the rectangle can be reordered modularly and* not *permuted. That is, given colors a and b, the fundamental domain a-b is equivalent to the fundamental domain b-a (where the fundamental domain is a rectangle (union of two squares), horizontal or vertical). Similarly for three squares, given colors a, b, and c, the fundamental domain a-b-c is equivalent to b-c-a is equivalent to c-a-b, and the fundamental domain a-c-b is equivalent to c-b-a is equivalent to b-a-c. We would refer to color a as the "lead color" in this case.*

Proof. If this observation isn't obvious at first, consider the band that a rectangular fundamental domain would make. If we look at the coloring encoded on this band, we will see *a*-*b*-*a*-*b*-. . . for two squares, and either *a*-*b*-*c*-*a*-*b*-*c*-. . . or *a*-*c*-*b*-*a*-*c*-*b*-. . . for three squares. But notice: the *a*-*b*-*a*-*b*-. . . is the same as a *b*-*a*-*b*-*a*-. . . pattern, the *a*-*b*-*c*-*a*-*b*-*c*-. . . is the same as the *b*-*c*-*a*-*b*-*c*-*a*-. .

. and the c-a-b-c-a-b-. . . pattern, and the a-c-b-a-c-b-. . . is the same as the c-b-a-c-b-a-. . . and the b-a-c-b-a-c-. . . pattern; it simply depends on where we begin our encoding. However, the encoding is infinitely long since we are in $\mathbb{R}^2$, so for every pattern there is another continuance of that pattern before and after it. This makes the choosing of the beginning color arbitrary, for the same pattern will yield the same tiling. Therefore we can just deem color a the "lead color" in order to encompass the various possible reorderings. $\square$

This eliminates the possibility of redundant tilings in a moduli space, and it allows us to connect different points of the moduli space that we might have overlooked. This becomes evident for the three-square moduli space.

With this proposition, we can now look at the four base tilings for the three-square case. Remember we will not include any redundant fundamental domains.

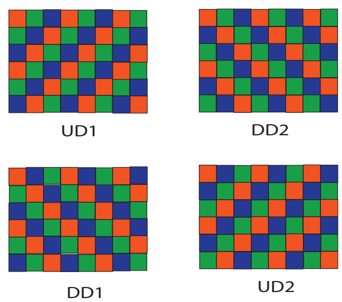

Figure 44: The four base tilings for tilings by three squares, named UD1, DD1, UD2, and DD2

Note we have labeled the base cases "UD1", "DD1", "UD2", "DD2" for ease of identification. Right away, we can see that we can deform DD1 to UD1 by a downward shift by the tall rectangular fundamental domain, and we can deform DD2 to UD2 by a downward shift by the tall rectangular fundamental domain.

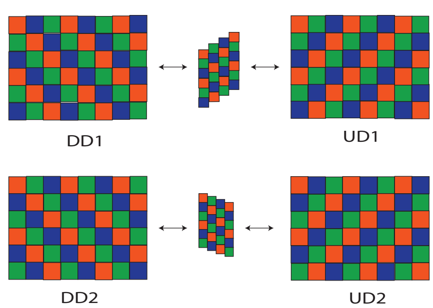

Figure 45: Deforming the base tilings by a downward shift of the tall rectangular fundamental domain

Note we also can deform DD2 to UD1 by a right shift by the wide rectangular fundamental domain, and we can deform DD1 to UD2 by a right shift by the wide rectangular fundamental domain.

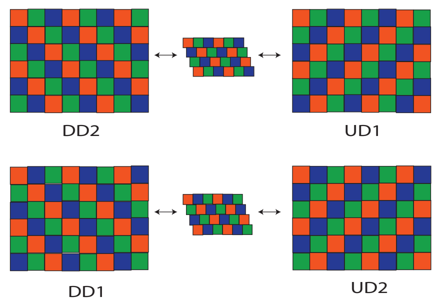

Figure 46: Deforming the base tilings by a right shift of the wide rectangular fundamental domain

But now note we can connect these four base tilings into a nice square figure, forming a more interesting pattern than the simple point we had as our base case in the one and two-square cases.

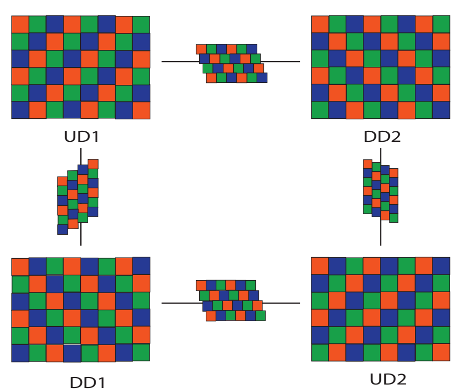

Figure 47: The four base tilings and the deformations that connect them

There is more shifting we can do, however. Let us return to the first case of deforming DD1 to UD1 by a downward shift by the tall rectangular fundamental domain. We can continue the deformation after reaching UD1, still shifting downward, until we reach a new line-up. In this tiling, which we will name "V1", we have a horizontal band of orange above a horizontal band of green above a horizontal band of blue above a horizontal band of orange again, and the pattern continues. We could continue the downward shift deformation we were originally doing, and this would complete the loop and bring us back to DD1, or we could also stay on V1 and shift the bands horizontally against each other like we did in the two-square case.

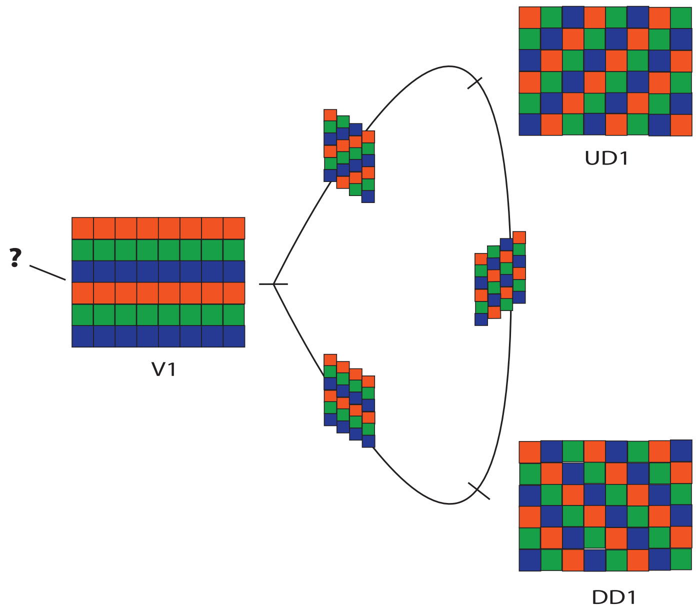

Figure 48: Continuing the downward shift deformation to get to the new tiling V1 and to complete a moduli space

As one may have guessed, we have free variables with these horizontal band shifts as in the two-square case, but here we have one more free variable than in the two-square case. We can fix the orange band (thereby fixing one of the translational vectors), then we can shift the green band independently, we can shift the blue band independently, and we can shift the next orange band independently, making the second translational vector free. This gives us three variables for the deformation space of the V1 tiling, which means we can represent this moduli space with a three-dimensional torus, *not* a two-dimensional torus or a loop.

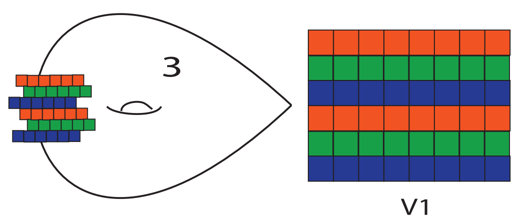

Figure 49: The 3-D toroidal moduli space on which V1 lies

This adds a new type of moduli space to our tiling system, and as one may guess it does not happen one time, but four times. We can continue the deformation of DD2 to UD2 to get the tiling “V2”, we can continue the deformation of DD2 to UD1 to get “H1”, and we can continue the deformation of DD1 to UD2 to get the tiling “H2”. With each of these, we can continue on the original deformation path to complete the loop, or we can branch off from V2, H1, and H2 into three-dimensional tori. We then get a very interesting and structured moduli space combining all of these spaces.

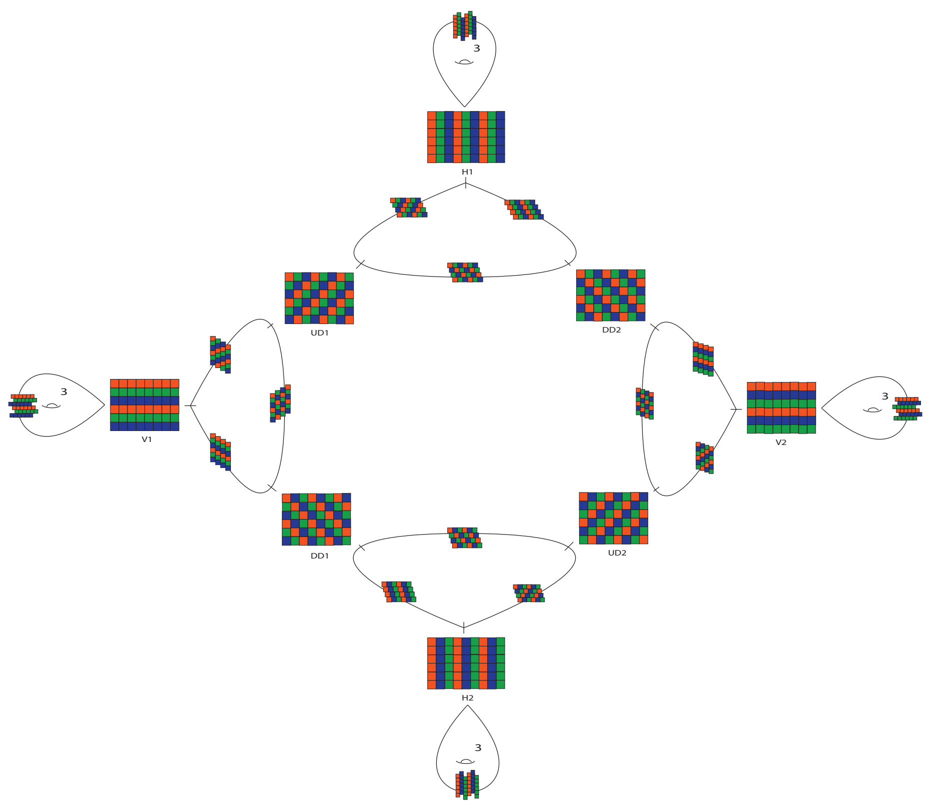

Figure 50: The main structure of the three square tiling system, which includes the four base tilings and the four tilings with bands (V1, H1, V2, H2)

This is only the beginning of the three-square tiling system, however. We still have yet to deform by independent scaling, which we will now do. As before with the shifts, we will have to look at the two main fundamental domains for each of the four base tilings, the vertical rectangle and the horizontal rectangle.

To find the new deformation spaces, we want to look at scaling one or two of the squares in the fundamental domain; that brings up the possibilities of growing one square, shrinking one square, growing two squares, and shrinking two squares. (The two-square scaling must be done uniformly with both of the squares, otherwise we will get a fundamental domain that doesn’t tile. The same goes for three-square scaling; they must all be scaled uniformly. But that is the trivial uniform scaling deformation that we cannot use, making the above four options the only four.) There is a way to reduce these four cases to two cases, however, making our hunt for new deformation spaces much easier; we just have to look at the cases relativistically.

We want to say that growing one square is the same deformation as shrinking two squares and that growing two squares is the same as shrinking one. We can show this is true; it is merely a difference of perspective. We will do an example that will be universal for all of the possible fundamental domains.

Let us grow the orange square, but look at the other two squares from the point of view of the orange square. The orange square may not realize he is growing, so to him, it looks like the blue and the green squares are shrinking, i.e. we are deforming by shrinking two squares at the same time we are deforming by growing one square. Since we do not account for uniform scaling, we can conclude that the two deformations are equivalent. Similarly, if we shrink the orange square and observe from his point of view, it looks like the green and the blue squares are getting bigger. That is, we are deforming by growing two squares at the same time we are deforming by shrinking one. Again, since we do not account for uniform scaling, we can conclude that the two deformations are equivalent. (Note that another way to make this evident is to fix the area of the three squares to add up to 1.)

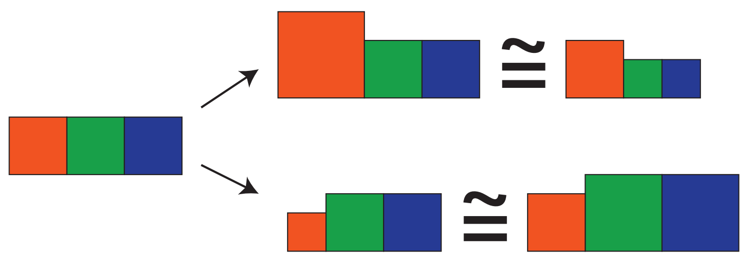

Figure 51: A diagram of the "relativistic" proof detailed above; note the cases are equivalent modulo scaling

This simplification means that we only have to worry about growing or shrinking a single square in the four base tilings. Unfortunately, this allows for twenty-four possible new deformation spaces, but, luckily, we can reduce this number.

The first and simpler deformation is shrinking one of the squares. This will lead us to a limit point and into the two-square moduli space. But first, recall Proposition 3.10: that color reordering yields the same tiling for colors a and b. From this, we can determine the equivalence of multiple tilings. As long as we have a "gridlock" tiling (like in the base cases, where everything is lined up), rectangular fundamental domains of the same orientation (either vertical or horizontal), and the same colors (e.g. blue followed by green and green followed by blue), then the tilings will be the same. The picture below gives an example:

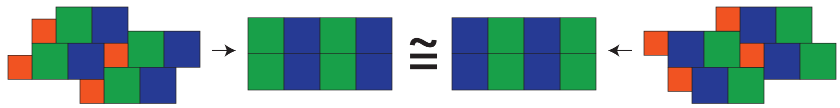

Figure 52: Note how different tilings can reach the same limit point

It would be helpful to define the term **gridlock** for ease of notation.

Definition 3.12. A periodic tiling by squares is called **gridlocked** if all the squares are lined up such that each vertex is touched by four squares. Necessarily, the squares are all of the same size.

Examples of gridlock tilings we have seen include the checkerboard tiling, the vertical band tiling, and the horizontal band tiling from the two-square case. We also have UD1, UD2, DD1, DD2, H1, H2, V1, and V2 from the three-square case. We know there are only those three gridlock tilings for the two-square case (since we know the entire moduli space for two squares), and we will see later that those eight tilings for the three-square case are the only gridlock tilings as well.

Note that at the limit point, the tiling will always be gridlocked (this will be explained combinatorially in the next section). Therefore the tiling will either be the checkerboard type, the vertical band type, or the horizontal band type. However, the previous example is indicative of the type of deformation to two squares we will always have, so the vertical or horizontal band tilings will always come from deforming the four base cases into two squares. Always remember that each point in the moduli space is unique and that the same tilings will have to occur at the same point, so we will have many instances where these limit points connect deformation spaces. We will see soon that this moduli space becomes too complicated and entangling to be accurately represented in a two-dimensional drawing, but we will try to make the diagrams as meaningful and readable as possible.

Let us first take UD1 with its vertical rectangular fundamental domain and look at shrinking the blue square, say to the left side to start. Notice that right away we get a perfect "L"-shaped fundamental domain. But notice DD1 has the same vertical rectangular fundamental domain, and that if we shrink the blue square there to the left side we get the same perfect "L"-shaped fundamental domain. Yet we know by Proposition 3.5 that if we have the same perfect "L"-shaped fundamental domain, we have the same tiling; therefore these two deformations must be represented by a single point in the moduli space. We call this a **shared deformation space** or **shared space** for short.

Definition 3.13. In a tiling system, a **shared deformation space** (or **shared space** for short), is a deformation space which is formed from similarly deforming two different tilings that share a common fundamental domain. The deformations of these two base tilings create the exact same deformed tiling, and the two base cases essentially "share" this deformation space.

This means for our moduli space, UD1 and DD1 must be infinitesimally close but not touching, yet they must be close enough to be connected by a point. In fact, they must be close enough to be connected by three points that do not touch (one for shrinking the blue square, one for shrinking the green square, and one for shrinking the orange square). Then from each of these points we can form a one-dimensional deformation space like we did with the "L"-shaped tiles for two squares. All three loops will reach a limit point into two squares, and the tilings will be horizontal band tilings.

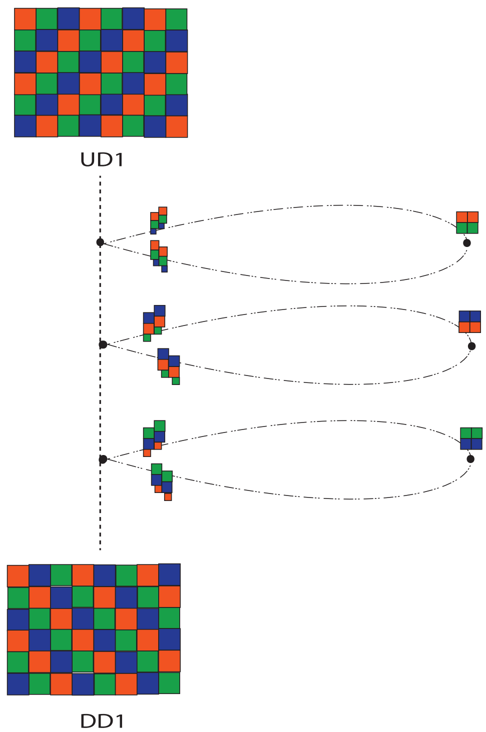

Figure 53: The limit points resulting from shrinking one square in the UD1 and DD1 tilings; note the short dashed lines to indicate a shared space

We can also do the same for the vertical fundamental domains of UD2 and DD2 since they are the same. Shrinking a square in UD2 yields the same tiling as shrinking the same color square in DD2, so, as in the above case, UD2 and DD2 must be infinitesimally close but not touching, and there must be three points connecting them. We can then branch off and form one-dimensional deformation spaces like before, and each loop will reach a limit point into two squares that yield horizontal band tilings.

But we just said a little while ago that horizontal or vertical band tilings composed of the same colors were identical. That means each pair of loops that has eliminated the same color must touch at the limit point, so we have six loops but only three limit points.

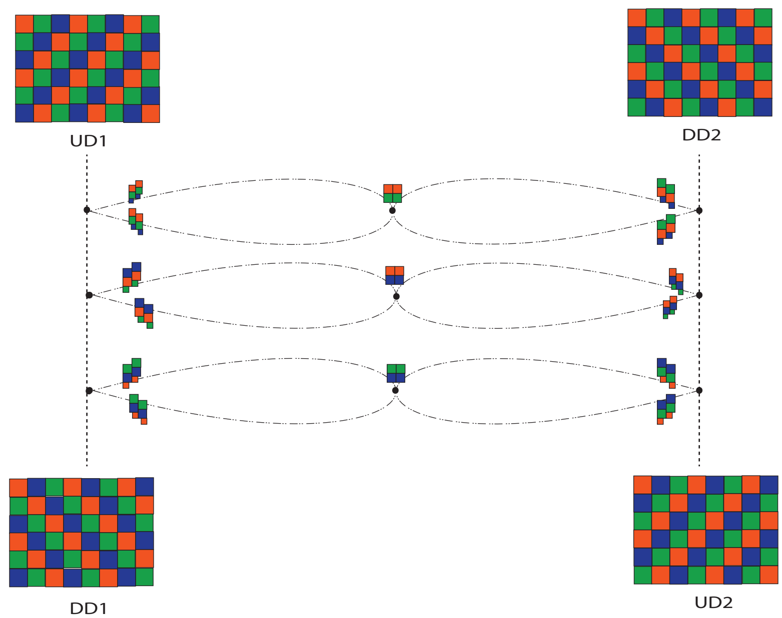

Figure 54: The equivalent limit points between the two pairs of base tilings sharing the same vertical fundamental domain

Now we can do the same process for the horizontal fundamental domains of the four base tilings. Note that UD1 and DD2 share the same horizontal fundamental domain, and DD1 and UD2 share the same one. This means, like above, UD1 and DD2 must be infinitesimally close and be connected by three points that have one-dimensional deformation spaces coming from them, and DD1 and UD2 must be infinitesimally close and be connected by three points that have one-dimensional deformation spaces coming from them. We get vertical band tilings at each limit point in this case, so like before, each pair of loops that has eliminated the same color must touch at the limit point, giving six loops but only three limit points.

The moduli space now gets very hard to draw effectively on a flat, two-dimensional surface, so we will use a different style line for the loops resulting from the vertical fundamental domains than for the loops resulting from the horizontal fundamental domains. Below is a preview of the drawing so that it will make sense later when put into the entire moduli space.

Recall that in the two-square moduli space, horizontal and vertical band tilings of the same color would eventually deform into each other. This means that out of our six limit points, the three pairs that have the same colors will connect in the two-square moduli space. We won't draw any lower square moduli spaces since the drawing is already complicated enough, but it is important to notice this fact.

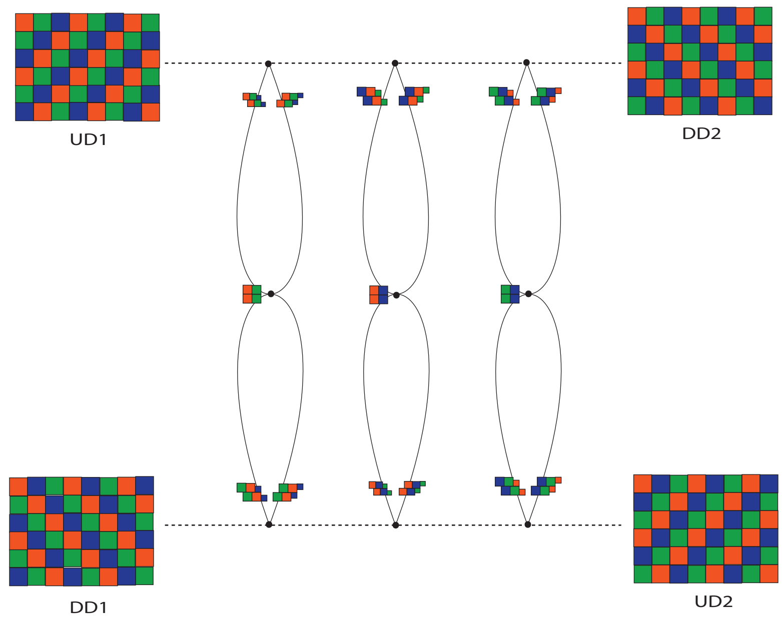

Figure 55: The equivalent limit points between the two pairs of base tilings sharing the same horizontal fundamental domain

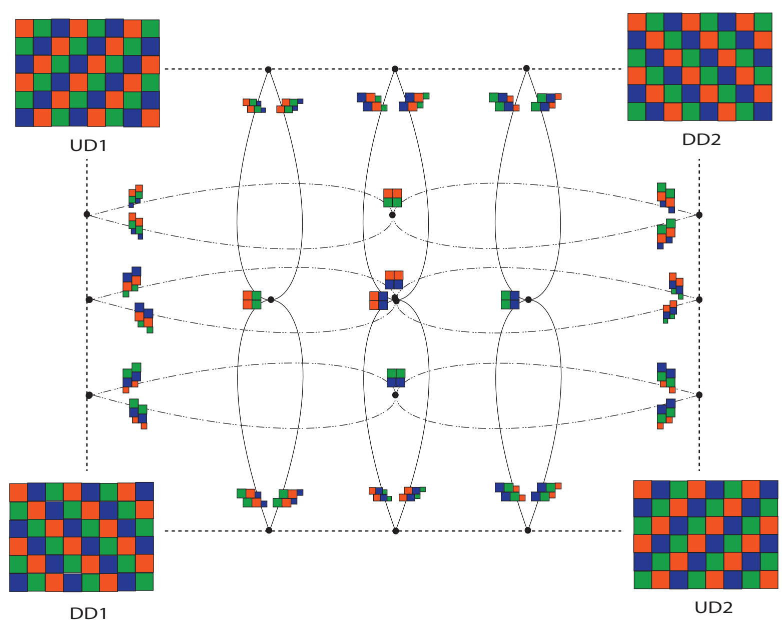

Figure 56: All one square shrinking deformations resulting from the four base cases

Now that we have the shrinking square case settled, we can look at what happens when we grow a square. First, let us look at growing the vertical fundamental domain of UD1 and DD1. Since they share the same vertical fundamental domain and any deformation by growing a square will turn the fundamental domain into a perfect "L"-shape, we again know by Proposition 3.5 that the two deformations produce the same tilings for each color square that is being grown. Therefore we have another three points connecting UD1 and DD1, and we can look at the deformation spaces for each.

Let us look at growing the orange square. Growing the other cases produce similar results, so this case will be indicative of the other two. First, let the orange square grow to the right, such that its left edge still matches up with the left edges of the green and blue squares. There will come a point in this growth where the orange square's side length is equal to the combined length of the side lengths of the blue and green square (which are of equal size). Note that this "L"-shaped tiling is one that is not perfect, for there exists another fundamental domain for this tiling (namely the rectangular one).

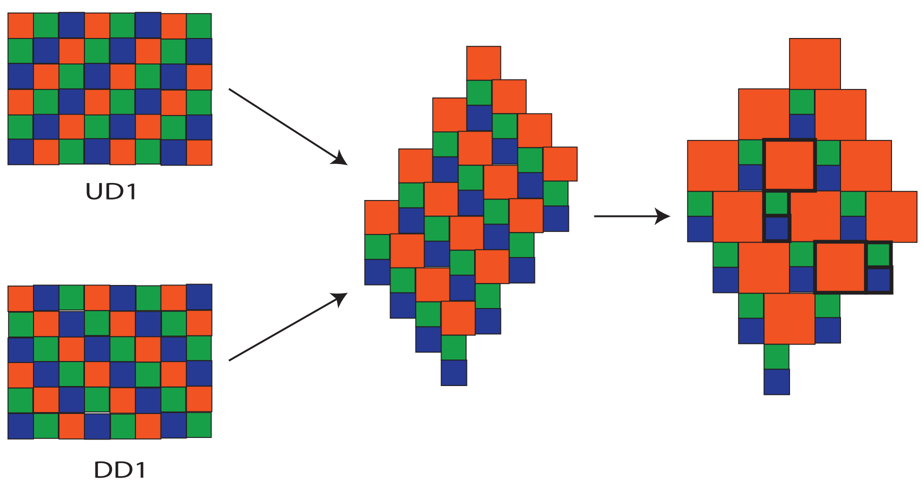

Figure 57: Growing the orange square from UD1 and DD1, making note of the particular tiling in which the three squares form a rectangular fundamental domain

There are two deformations we could take from this point. One involves shifting and the other involves a continuation of the growing we have done. We will come back to the shift and will continue on with the growing.

Recall growing one square is the same as shrinking two, so if we continue growing the orange one, we are constantly shrinking the green and the blue one (which are the same size). This means that this deformation will end in a limit point to one square, which is our first three-to-one square limit point.

Now we can grow the other two squares back from the other side of the orange one to complete the deformation space. We will go back through the

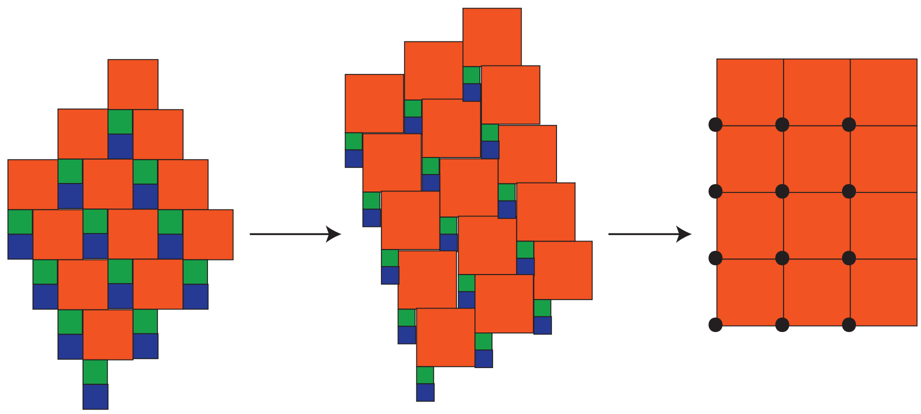

Figure 58: Growing the orange square and reaching a three-to-one square limit point

tiling with the non-perfect “L”-shaped fundamental domain and return to our starting place at UD1 and DD1. Note that the non-perfect “L”-shaped tiling is slightly different this time. As one may guess, the two non-perfect “L”-shaped tilings will be connected.

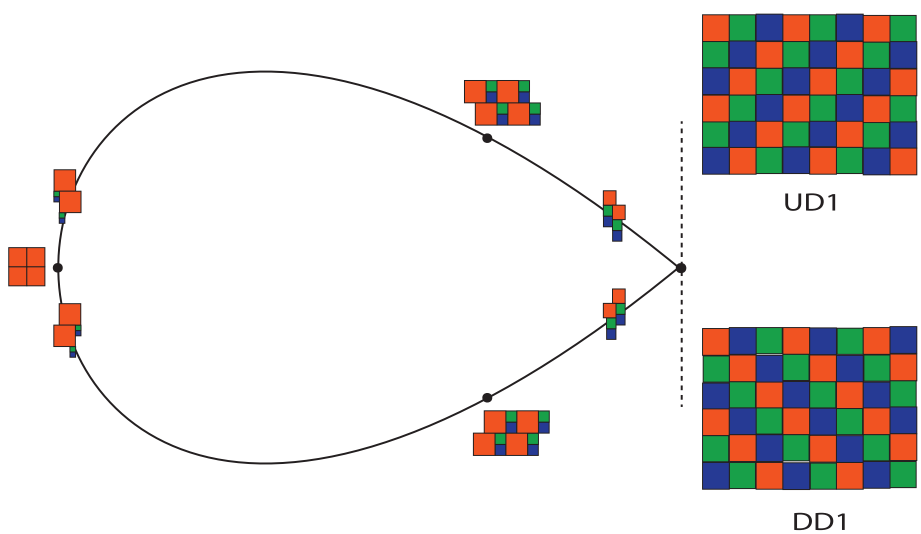

Figure 59: The entire single loop moduli space of growing one square

Let us return to the non-perfect “L”-shapes. Recall that we can use the shift as a deformation by shifting the horizontal bands of rectangles. If we look at the first of these tilings that we encountered, we can shift to the right or left (both will be the same but the involve moving around the moduli space in a different direction). Let us shift to the right.

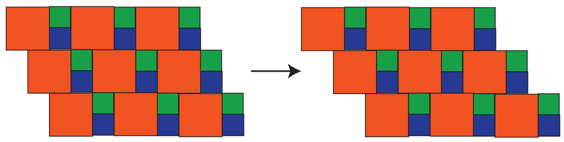

Figure 60: Shifting the rectangular fundamental domain to the right

There will come a point where the rectangular tiles are all aligned; the orange squares line up in vertical bands, and the green and blue squares line up in an alternating vertical band. This is a special case that we will discuss after we complete the shifting deformation which we are now on.

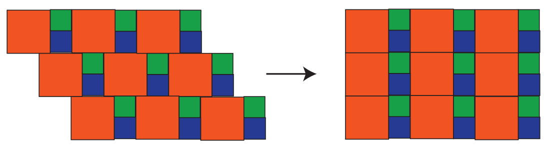

Figure 61: Further shifting the rectangular fundamental domain to the right, reaching a unique point in which we have vertical bands of squares

If we continue with the right shift, we will eventually reach the second non-perfect "L"-shape tiling we discovered in the above square-growth deformation. This is the connection between the two that was mentioned earlier. Now we can simply continue this right shift, and we will return back to the tiling at which we started.

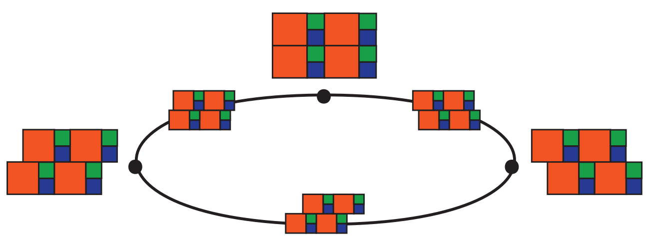

Figure 62: The moduli space of shifting the rectangular fundamental domain

Now we can look at the special rectangular tiling we just mentioned, where the orange squares are lined up in vertical bands and the green and blue squares line up in an alternating vertical band. This is the final barrier that stands between us and completing the three-square tiling system. Notice right away that it is possible to shift the two vertical bands independently, almost exactly like we did in the two-square case with vertical bands. As before, this will form a two-dimensional deformation space that is represented by a 2-D torus.

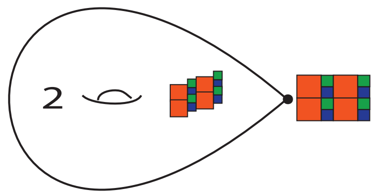

Figure 63: The special rectangular fundamental domain with the 2-D torus to which it is attached; note the example of a tiling that lies inside this torus

There are some special cases that arise in this torus, however. Notice if we only shift the alternating green and blue vertical band one square length up, we have a different rectangular fundamental domain than we started with. We could then perform a horizontal rectangle shift to the non-perfect "L"-shape, then shrink the orange square, and we would end up in the UD2 and DD2 base cases instead of UD1 and DD1. This tells us that the 2-D torus overlaps with the opposite one, connecting the two opposing sides in a new way. We will illustrate this important observation soon, but there are still two more special cases on the torus that we must address. Both of the cases involve breaking the horizontal bands and lining the vertical ones up in a certain way. Below are the two cases; we will call the first Case 1 and the second Case 2:

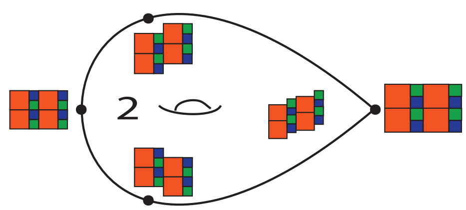

Figure 64: The two special cases on the 2-D torus; Case 1 is located near the top of the figure, while Case 2 is located near the bottom

Notice how in either case, two different rectangular fundamental domains arise depending on if you look at the orange square being on the left or right of the other two.

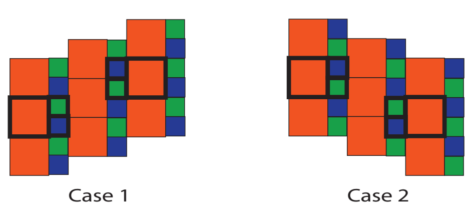

Figure 65: The Case 1 tiling and the Case 2 tiling; note the two different fundamental domains in each one

These two special alignments allow us to perform one final deformation that will let us complete the tiling system for three squares. There exists a tiling by three squares in which all squares are different sizes, but there are a few restrictions. The first is that the sum of the side lengths of the two smaller squares must be the same as the entire side length of the larger square. The next (which essentially follows from the first) is that the tile must be a perfect "L"-shape.

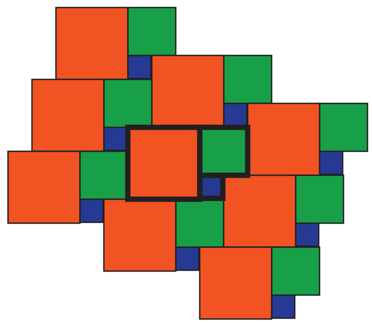

Figure 66: An example of a tiling by three squares in which all three squares are different sizes

We will see in the next section how the first restriction arises mathematically, but for now we will just perform the deformation. Let us take Case 1. First we will grow the green square while shrinking the blue one, and we will take the fundamental domain that places the orange square on the left of the other two.

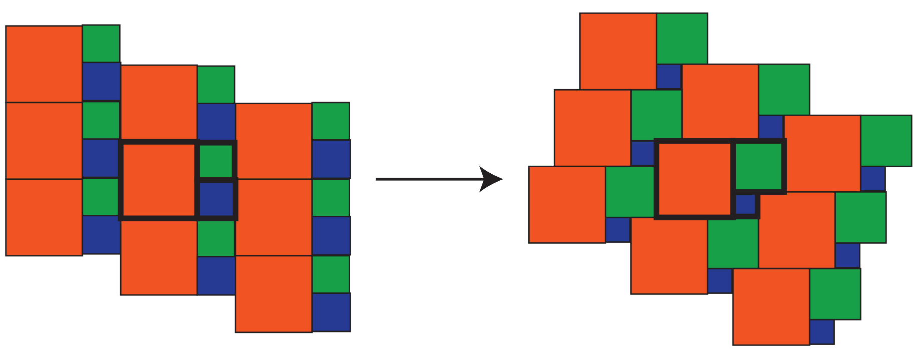

Figure 67: The beginning of the deformation starting from Case 1 in which we grow the green square while shrinking the blue square

If we continue this deformation of growing green while shrinking blue, we will eventually reach a limit point with the blue square disappearing to a point, leaving the orange and the green square. However, notice the pattern through which we reach the limit point:

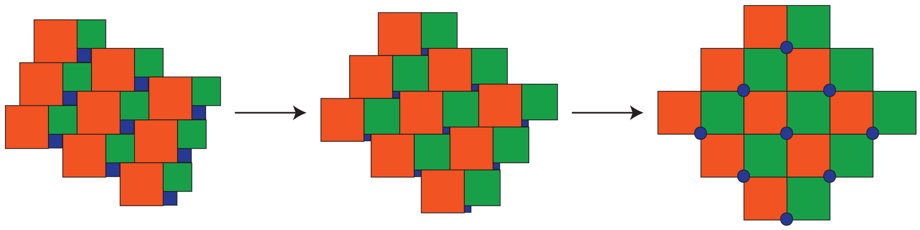

Figure 68: Continuing the above deformation, this time shrinking the blue square to a point

Notice that the squares do not line up in vertical bands like they did when we shrunk one square. Instead, they form the checkerboard pattern that is at the center of the two-square tiling system. This means that we have three points in the three-square tiling system that reduce to three different cases in the two-square system. We can think of it as an "underground network" for the two-square moduli spaces; they act as shortcuts to different parts of the three-square moduli space, creating a much more interesting link between the three-square and the two-square system than we had with the two-square and the one-square system.

Now we must continue the deformation. Since we already shrunk the blue square to a point, we must bring it back on the other side like we have done in the past. As before, we do this by taking a vertical mirror image of what we just did. Continuing to grow the blue square and shrink the green, we end up at the Case 2 tiling, not back at our original starting point.

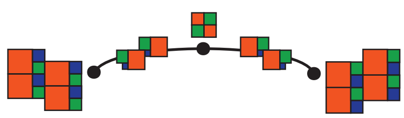

Figure 69: The half loop that we get from the deformation in which the green square is larger than the blue square

We can do the same thing again for the Case 1 tiling if we grow the blue square while shrinking the green. This time at the limit point we will have an orange and blue square checkerboard pattern. Continuing the deformation, we again end up at the Case 2 tiling, not at our original starting point.

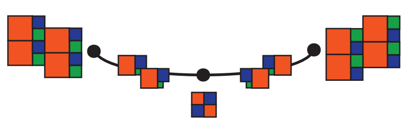

Figure 70: The half loop that we get from the deformation in which the blue square is larger than the green square

But now note that we have already taken care of the deformations that come from the Case 2 tiling. This allows us to complete the internal structure of the 2-D toroidal moduli space.

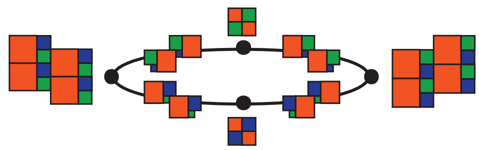

Figure 71: The internal loop of the 2-D torus

Notice the two checkerboard tilings at the two reduction points. We will see that these exact same tilings come up multiple times throughout the entire tiling system. For the ease of illustrating the moduli spaces on a two-dimensional page, we will not connect them but instead mark each point by a two-color star. This two-color star will stand for the checkerboard pattern with two squares that have the star's colors. Any stars that are the same are the same point. This will help us in decreasing the overwhelming crowdedness and complexity of the three-square system, and it will better allow us to see how the spaces connect. We will do the same for the one-square case. Instead of connecting all one-square limit points of the same color, we will just mark those points with a solid-colored star of the square's color.

Figure 72: The various stars we will use in the three-square moduli space; the two-color stars stand for checkerboard tilings while the solid-colored stars stand for gridlock tilings by the same-colored square

Now we can combine this information to get a clear picture of what is going on in the 2-D torus that we have looked at so deeply:

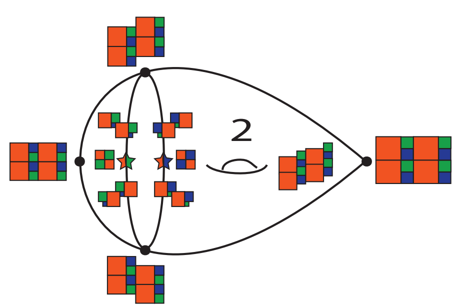

Figure 73: The 2-D torus moduli space along with its internal structure

It is important to remember that the 2-D torus is shared between opposite sides (e.g. the UD1/DD1 side and the UD2/DD2 side). When illustrating the entire moduli space, we will not connect the two since it would make our picture impossible to read. Instead, we will indicate a copy of the 2-D torus with a dashed line. We will still mark the two tilings that join the opposing sides, but it will be made clear that there is only one of that type of 2-D torus.

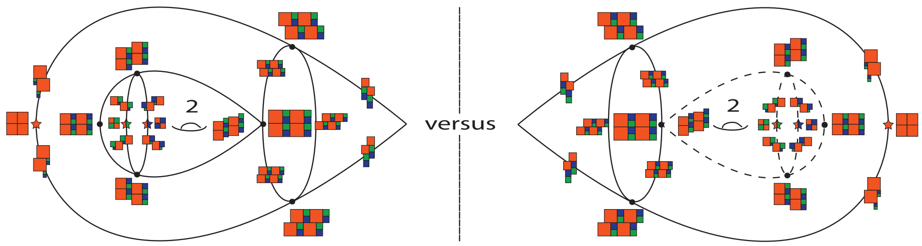

Figure 74: An example of the 2-D torus (and the surrounding moduli space) on both sides of the entire three-square moduli space; note the dashed lines on the torus to the right to indicate that it is repeated

Now that we have the final piece figured out, we can repeat the deformation for each color in the fundamental domain. Notice that this repetition will create many more two-color stars, indicating the same two-square checkerboard patterns at limit points. For the UD1 and DD1 starting case, we still must grow the green square, and we must grow the blue square. Completing those two tasks will yield the picture depicted below.

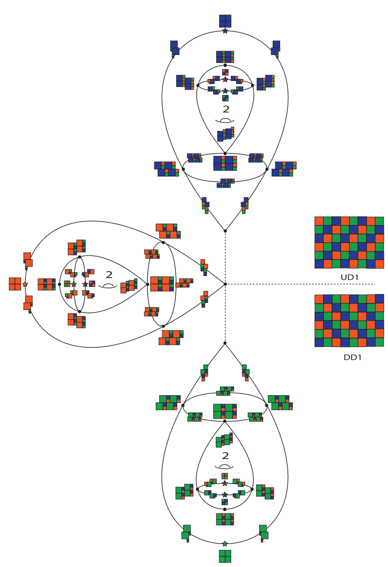

Figure 75: The complete one-square growth for the UD1 and DD1 starting case

We can also complete the opposing UD2/DD2 starting case since the two sides share tori. They also share three-to-one square limit points, so we can now make use of the single colored stars. (As one may surmise, the single colored stars will occur on each side.) They share three-to-two limit points as well (and so do all of the other sides).

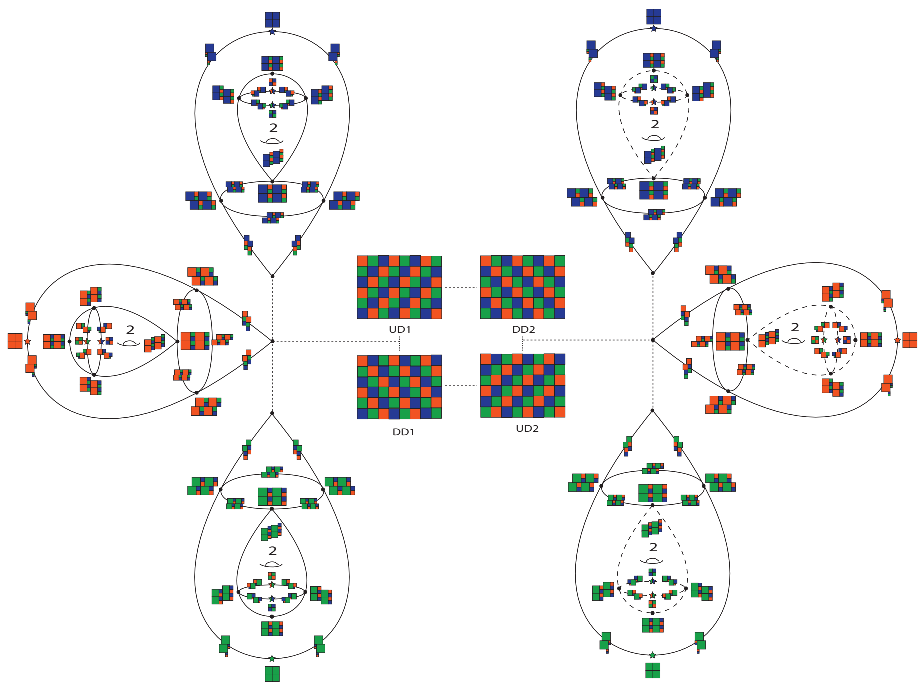

Figure 76: The complete one-square growth for the UD1 and DD1 starting case along with the UD2 and DD2 starting case; note the dashed lines for the 2-D tori on the right side of the figure to indicate they are repeated

Now we only have the horizontal fundamental domains from the four base cases to worry about. However, it is all the same idea, only with rectangles that are tall instead of wide. We apply everything we have discussed with this “one growing square” deformation, and we finally lay eyes upon our long sought after prize: the complete moduli space for the three-square tiling system.

The end result is a beautiful topological object. Though it is hard to effectively represent on a two-dimensional surface, the moduli spaces and the connections are evident. The symmetry that this tiling system creates is unbelievable. After examining/admiring the three-square system for a little while, we now must focus in on the most important question: why do these tiling systems look the way they do? What is the mathematics behind it all?

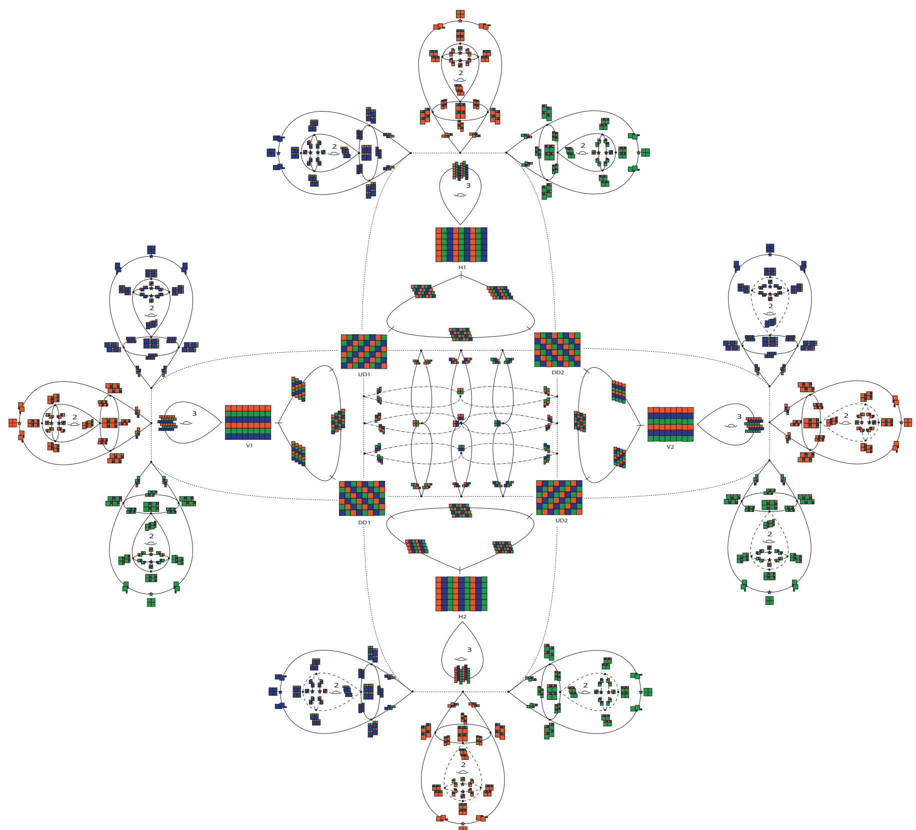

Figure 77: The entire three-square moduli space

3.3 Combinatorial Structure

We now know what each tiling by one, two, and three squares looks like, and we know how their deformation spaces connect, but it is important to understand *why* they look the way they do and deform the way they do. To do this, we use what is called the **dual graph** of a tiling.

Definition 3.14. The **dual graph** of a tiling is a graph we place over the entire tiling to look at how each individual square interacts with its neighboring squares. First, we draw a vertex in the middle of each square and label the vertex with the size of the square (say a, b, or c, since we are only concerned with three different squares in our case). Then, given a square, we look at what is touching its top and bottom edges and what is touching its left and right edges. For each square touching our square's top or bottom edges, we draw a colored line (say red) to that neighboring square to indicate a vertical edge, and for each square touching our square's left or right edges, we draw a different colored line (say yellow) to that neighboring square to indicate a horizontal edge. Note that squares that touch at a vertex do not qualify as neighbors. Also note that the vertical and horizontal edges naturally make the graph a directed graph. We repeat this process for all squares in the tiling to obtain the dual graph.

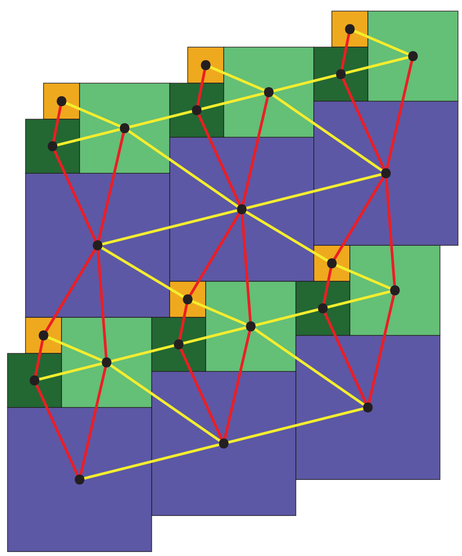

Figure 78: An example of a dual graph

So why introduce this dual graph? One of the things that the dual graph can tell us is how we can deform a tiling and how many different deformation spaces intersect the tiling. Let us first look at deforming the tiling, then we will discuss the deformation space intersections.

What we want to do now is attach data to the dual graph. We have already labeled the vertices by square size, so the next thing we do is "reduce" the dual graph to avoid any redundant edges. Let us think of these edges as vectors, directing us from one vertex to another. The only constraint we will put on the vectors is that they all point in the positive x-direction. If this does not determine the y-direction, then we will require the vector to point in the positive y-direction as well. Now for the reduction: we need to look at a single fundamental domain and the pieces around it (dual graph still intact). Now we must label each edge as a vector according to our above constraints. For example, we can have a vector going from square a to square a and one from square a to square b (where a and b simultaneously denote the square's size and name). Note that if we have a square a to square a vector, all other vectors that direct square a to a translate of itself will be equivalent, i.e. there is only one path from square a to square a. The a to b vector however, will not be the same as the b to a vector as long as we follow our vector constraints above. We continue on like this, labeling the dual graph edges as vectors and making sure we do not count the same vector twice (modulo translation). The process is harder to describe in words than to show, so below is a generic example.

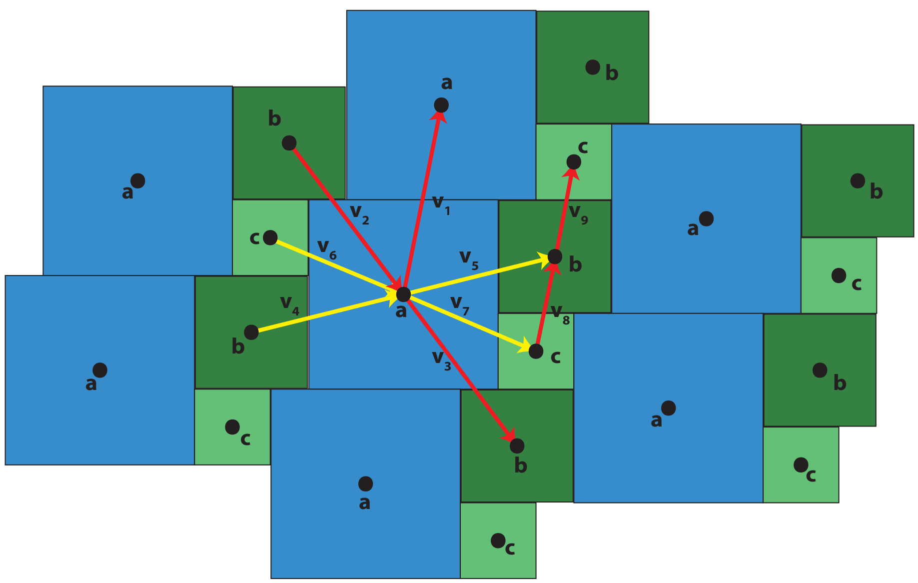

Figure 79: An example of a tiling by three squares with a "reduced" dual graph, in which all three squares are labeled and all nine vectors are labeled

The next step is to explicitly define each of the vectors we have just labeled. Let us continue with the above example, as it helps us understand the process better. Looking at v_1, we will notice that in the vertical direction, it moves from the vertex of square a to another vertex of square a, so it travels $2a$ vertically. (We will say the size a, b, or c is half of the square's side length to avoid unnecessary fractions in our vectors.) We can see that the vector moves some unidentified amount in the horizontal direction, so we call this value x_1 and get $v_1 = (x_1, 2a)$.

For v_2, note that it moves by a variable amount x_2 horizontally and moves downward $a + b$, so we get $v_2 = (x_2, -(a + b)) = (x_2, -a - b)$

One could easily identify the remaining vectors, calling unknown horizontal movements x_i and unknown vertical movements y_i for vector v_i, and we find the following values for each vector: $v_3 = (x_3, -a - b)$, $v_4 = (a + b, y_4)$, $v_5 = (a+b, y_5)$, $v_6 = (a+c, -y_6)$, $v_7 = (a+c, -y_7)$, $v_8 = (x_8, b+c)$, and $v_9 = (x_9, b+c)$.

Now, how can we find these x_i and y_i values? The answer lies in the loops composed of three (or sometimes four) vectors that the dual graph holds. With each of these loops, we can create a vector equation that equals zero, and with all of these equations, we have a system that allows us to solve for the unknown values. But, as before with the vectors, we cannot count the same loop twice. That is, loops composed of the same vectors located at the same places in the loop are equivalent (modulo translation). We will label the loops with Roman numerals, and we will continue on with our above example.

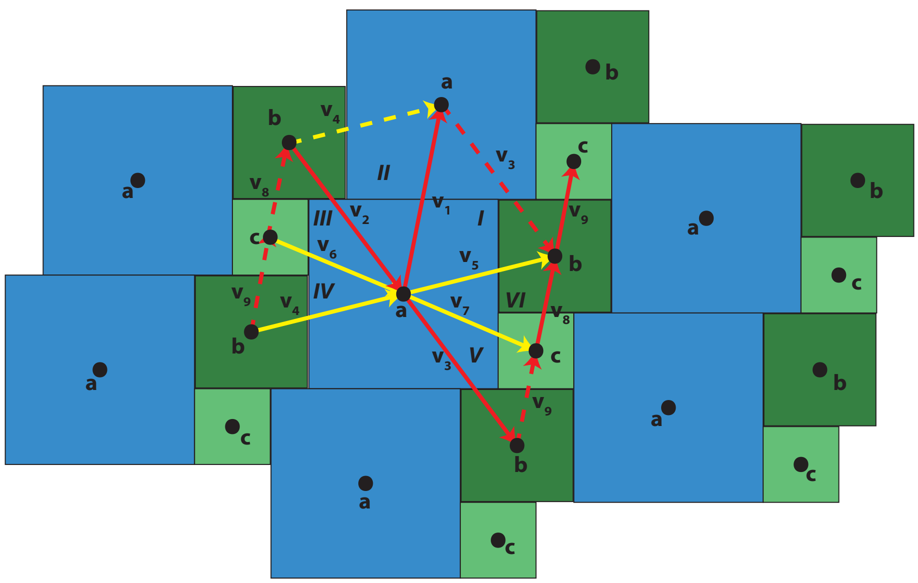

Figure 80: All of the necessary data attached to the tiling; note the dashed vectors indicate a vector that is copied by translation in order to complete a loop

In this case we find six loops, which we label with Roman numerals. The six equations are:

$$\begin{aligned}
&I : v_5 - v_3 - v_1 = 0\\
&II : v_1 - v_4 + v_2 = 0\\
&III : v_6 - v_2 - v_8 = 0\\
&IV : v_4 - v_6 - v_9 = 0\\
&V : v_9 - v_7 + v_3 = 0\\
&VI : v_8 - v_5 + v_7 = 0
\end{aligned}$$

Note that we can write any one equation in terms of the other five (e.g. $VI = -I - II - III - IV - V$), so the equations are linearly dependent. (In general, the equations are dependent because together they sum to zero; this means we can always write one equation in terms of the others.) This allows us to eliminate one equation and lets us not use any redundant information. We will toss out equation VI, but in cases with more squares, it would help to carefully choose which equation to throw out to make the linear algebra much easier to do.

For each equation, we get two separate equations: one for the horizontal movement, and one for the vertical movement. We will use the subscript 1 for horizontal movement and the subscript 2 for vertical movement for ease of notation. So now we have ten equations, and they are as follows:

$$\begin{aligned}
&I_1 : a + b - x_3 - x_1 = 0\\
&I_2 : y_5 + a + b - 2a = 0\\
&II_1 : x_1 - a - b + x_2 = 0\\
&II_2 : 2a - y_4 - a - b = 0\\
&III_1 : a + c - x_2 - x_8 = 0\\
&III_2 : -y_6 + a + b - b - c = 0\\
&IV_1 : a + b - a - c - x_9 = 0\\
&IV_2 : y_4 + y_6 - b - c = 0\\
&V_1 : x_9 - a - c + x_3 = 0\\
&V_2 : b + c + y_7 - a - b = 0
\end{aligned}$$

Now all we do is solve each equation. What we want to do in this case is get each x_i and y_i in terms of a, b, and c, and then check for the constraints on a, b, and c. First we want to look at the equations where we only have a single x_i or a single y_i. This includes equations I_2, II_2, III_2, IV_1, and V_2, and we find that $y_5 = a - b$, $y_4 = a - b$, $y_6 = a - c$, $x_9 = b - c$, and $y_7 = a - c$, respectively, from each of these equations.

Next we want to find an equation where we know one of the x_i or y_i quantities in terms of a, b, and c, but don't know the other. Equation V_1 is the case for which we are looking, for we know the quantity x_9 and now can find the quantity x_3. We get $x_3 = a - b + 2c$.

Now we simply continue this process, looking for the equations with one known variable and one unknown variable. We can now find x_1 from I_1, then we can find x_2 from II_1, and finally x_8 from III_1. We find $x_1 = 2b - 2c$, $x_2 = a - b + 2c$, and $x_8 = b - c$.

Notice how we never used equation IV_2. This equation tells us the constraints on the square size variables a, b, and c. If we plug in y_4 and y_6 into IV_2, we find $2a - 2b - 2c = 0$, or $2a = 2b + 2c$. Recall that we used a, b, and c as half of the side length for each square to avoid fractions in our equations. This means $2a$, $2b$, and $2c$ indicate the entire side length for each square. This final constraint tells us that the smaller squares b and c must have their side lengths always sum to the larger square a's side length.

Now, how does this relate to the deformation space? The last constraint tells us most of the information we need to know. Since we don't count scaling, the size a will be given, and we can only choose sizes b or c. But if we choose one, the other one will automatically be given by the size constraint above, so we only get to freely choose one of the sizes. Furthermore, since all other variables are in terms of a, b, and c, this free choice is the only free choice we get with this space of tilings. That means we only have one free variable with this set up, which means we have a one-dimensional deformation space and therefore a loop for the moduli space, all of which we predicted in the last section. The use of the dual graph and the named vectors and loops gives us the information we need on how we can deform a tiling, though in some cases this deformation space may be easier to recognize on some tilings than on others (the above case was pretty simple).

Before moving on, note how there were three squares, six loops, and nine vectors. This is no coincidence; it turns out this is directly related to the Euler characteristic for *tori*. (It makes sense that we look at tori since our fundamental domain repeats itself on all sides.)

Recall that the Euler characteristic $\chi = V - E + F$, where V denotes vertices, E denotes edges, and F denotes faces. For tori, we know that $\chi = 0$, always (see Matveev's entry in the Encyclopedia of Mathematics [2]). This means $E = V + F$. We already know V, the vertices, since each squares' center represents a vertex in the dual graph, so we have three of them. Since edges simply connect vertices, it makes sense that our nine vectors would be the edges that we want. Going one step further, faces connect edges, so our loop equations represent the faces in this case. (Normally faces would be two-dimensional objects, but in our case we created loops to represent each of these two-dimensional objects to allow us to use algebra to solve for the unknowns. Though the dimensions are not the same, the numbers are, and that is what we are concerned about.)

We can generalize this relationship, writing E and F in terms of V. Given a set of vertices V, we will have $3V$ edges connecting each of the vertices, and we will have $2V$ faces (or loops) connecting each of the edges. That means the original equation $E = V + F$ is transformed into $3V = V + 2V$, which is still true. This is the relationship that will always hold in the dual graph analysis of the tiling, so if one applies the analysis and finds that the squares, vectors, and loops don't make this formula true, there is a mistake. This formula turns out to be very useful to ensure we are on the right track with trying to find the dimension of the deformation space.

Below are more examples in which we find the dimension of a deformation space using the dual graph. Note we can analyze cases in which there are more than three squares, and the procedure still works.

Example 3.15. We begin by putting a "reduced" dual graph on the tiling and labeling all squares, vectors, and loops.

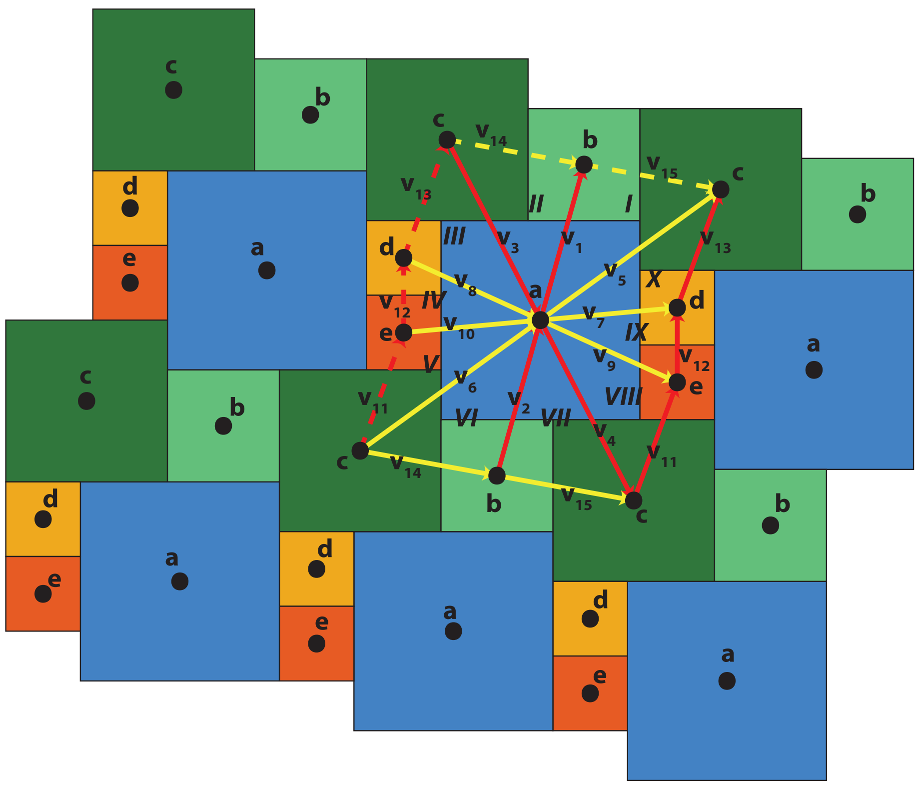

Figure 81: A tiling by five squares with all squares, vectors, and loops labeled

We next define each of the vectors in the picture.

$$v_1 = \begin{pmatrix} x_1 \\ a+b \end{pmatrix} \quad v_2 = \begin{pmatrix} x_2 \\ a+b \end{pmatrix} \quad v_3 = \begin{pmatrix} x_3 \\ -a-c \end{pmatrix} \quad v_4 = \begin{pmatrix} x_4 \\ -a-c \end{pmatrix}$$

$$v_5 = \begin{pmatrix} a+c \\ y_5 \end{pmatrix} \quad v_6 = \begin{pmatrix} a+c \\ y_6 \end{pmatrix} \quad v_7 = \begin{pmatrix} a+d \\ y_7 \end{pmatrix} \quad v_8 = \begin{pmatrix} a+d \\ y_8 \end{pmatrix}$$

$$v_9 = \begin{pmatrix} a+e \\ y_9 \end{pmatrix} \quad v_{10} = \begin{pmatrix} a+e \\ y_{10} \end{pmatrix} \quad v_{11} = \begin{pmatrix} x_{11} \\ c+e \end{pmatrix} \quad v_{12} = \begin{pmatrix} x_{12} \\ d+e \end{pmatrix}$$

$$v_{13} = \begin{pmatrix} x_{13} \\ c+d \end{pmatrix} \quad v_{14} = \begin{pmatrix} b+c \\ y_{14} \end{pmatrix} \quad v_{15} = \begin{pmatrix} b+c \\ y_{15} \end{pmatrix}$$

Now we write down the equation for each loop. Note we have ten equations in this example.

$$\begin{aligned}
&I : v_5 - v_{15} - v_1 = 0 \\
&II : v_1 - v_{14} + v_3 = 0 \\
&III : v_8 - v_3 - v_{13} = 0 \\
&IV : v_{10} - v_8 - v_{12} = 0 \\
&V : v_6 - v_{10} - v_{11} = 0 \\
&VI : v_{14} + v_2 - v_6 = 0 \\
&VII : v_{15} - v_4 - v_2 = 0 \\
&VIII : v_{11} - v_9 + v_4 = 0 \\
&IX : v_{12} - v_7 + v_9 = 0 \\
&X : v_{13} - v_5 + v_7 = 0
\end{aligned}$$

Since the equations are linearly dependent, we can toss one out (say equation X). Then we can split up the remaining equations into their x and y components, leaving us with eighteen equations in total.

$$\begin{aligned}
&I_1 : a + c - b - c - x_1 = 0 \\
&I_2 : y_5 - y_{15} - a - b = 0 \\
&II_1 : x_1 - b - c + x_3 = 0 \\
&II_2 : a + b - y_{14} - a - c = 0 \\
&III_1 : a + d - x_3 - x_{13} = 0 \\
&III_2 : y_8 + a + c - c - d = 0 \\
&IV_1 : a + e - a - d - x_{12} = 0 \\
&IV_2 : y_{10} - y_8 - d - e = 0 \\
&V_1 : a + c - a - e - x_{11} = 0 \\
&V_2 : y_6 - y_{10} - c - e = 0 \\
&VI_1 : b + c + x_2 - a - c = 0 \\
&VI_2 : y_{14} + a + b - y_6 = 0 \\
&VII_1 : b + c - x_4 - x_2 = 0 \\
&VII_2 : y_{15} + a + c - a - b = 0 \\
&VIII_1 : x_{11} - a - e + x_4 = 0 \\
&VIII_2 : c + e - y_9 - a - c = 0 \\
&IX_1 : x_{12} - a - d + a + e = 0 \\
&IX_2 : d + e - y_7 + y_9 = 0
\end{aligned}$$

In this example we want to find all x_i and y_i in terms of a, b, c, d, and e like we did in the previous example. The first set of equations we can do this for is I_1, II_2, III_2, IV_1, V_1, VI_1, VII_2, $VIII_2$, and IX_1. These equations reduce to the following:

$$\begin{aligned}
&I_1 \rightsquigarrow x_1 = a - b\\
&II_2 \rightsquigarrow y_{14} = b - c\\
&III_2 \rightsquigarrow y_8 = d - a\\
&IV_1 \rightsquigarrow x_{12} = e - d\\
&V_1 \rightsquigarrow x_{11} = c - d\\
&VI_1 \rightsquigarrow x_2 = a - b\\
&VII_2 \rightsquigarrow y_{15} = b - c\\
&VIII_2 \rightsquigarrow y_9 = d - a\\
&IX_1 \rightsquigarrow x_{12} = d - e
\end{aligned}$$

Note right away we can combine equations IV_1 and IX_1 to find:

$$x_{12} = d - e = e - d \Rightarrow \{x_{12} = 0 \text{ and } d = e\}.$$

Now we can combine what we found in the above nine equations with I_2, II_1, IV_2, VI_2, VII_1, and IX_2 to find six more x_i and y_i values.

$$\begin{aligned}
&I_2 \rightsquigarrow y_5 - b + c - a - b = 0 \Leftrightarrow y_5 = a + 2b - c\\
&II_1 \rightsquigarrow a - b - b - c + x_3 = 0 \Leftrightarrow x_3 = 2b + c - a\\
&IV_2 \rightsquigarrow y_{10} - d + a - d - d = 0 \Leftrightarrow y_{10} = 3d - a\\
&VI_2 \rightsquigarrow b - c + a + b - y_6 = 0 \Leftrightarrow y_6 = a + 2b - c\\
&VII_1 \rightsquigarrow b + c - x_4 - a + b = 0 \Leftrightarrow x_4 = 2b + c - a\\
&IX_2 \rightsquigarrow d + d - y_7 + d - a = 0 \Leftrightarrow y_7 = 3d - a
\end{aligned}$$

With the values obtained from these equations, we can write equations V_2 and $VIII_1$ using only a's, b's, c's, and d's.

$$\begin{aligned}
&V_2 \rightsquigarrow a + 2b - c - 3d + a - c - d = 0 \Leftrightarrow 2a + 2b = 2c + 4d \Leftrightarrow a + b = c + 2d\\
&VIII_1 \rightsquigarrow c - d - a - d + 2b + c - a = 0 \Leftrightarrow 2a - 2b = 2c - 2d \Leftrightarrow a - b = c - d
\end{aligned}$$

We can now solve for x_{13} located in III_1 using the x_3 value we found in II_1 and the constraints on a, b, c, and d we found in $VIII_2$.

$$\begin{aligned}
III_1 &\rightsquigarrow a + d - 2b - c + a - x_{13} = 0 \Leftrightarrow 2a - 2b - c + d = x_{13}\\
&\Leftrightarrow 2c - 2d - c + d = x_{13} \Leftrightarrow x_{13} = c - d
\end{aligned}$$

Now let us look at combining V_2 and $VIII_1$ in order to get more explicit constraints on the square sizes. We already know $d = e$, so we take $V_2 - VIII_1$ to get:

$$V_2 - VIII_1 : (a + b = c + 2d) - (a - b = c - d) \rightsquigarrow 2b = 3d \Leftrightarrow b = \frac{3}{2}d$$

Now we take $V_2 + VIII_1$ to get:

$$V_2 + VIII_1 : (a + b = c + 2d) + (a - b = c - d) \rightsquigarrow 2a = 2c + d \Leftrightarrow c = a - \frac{d}{2}$$

With these new constraints, we claim that a and d are free variables, and we will show this by explicitly writing each variable in terms of a and d.

$$\begin{aligned}
&a = a\\
&b = \frac{3}{2}d\\
&c = a - \frac{d}{2}\\
&d = d\\
&x_1 = x_2 = a - \frac{3}{2}d\\
&x_3 = x_4 = 2(\frac{3}{2}d) + (a - \frac{d}{2}) - a = 3d - \frac{d}{2} = \frac{5}{2}d\\
&y_5 = y_6 = a + 2(\frac{3}{2}d) - (a - \frac{d}{2}) = \frac{7}{2}d\\
&y_7 = y_{10} = 3d - a\\
&y_8 = y_9 = d - a\\
&x_{11} = x_{13} = (a - \frac{d}{2}) - d = a - \frac{3}{2}d = x_1 = x_2\\
&x_{12} = 0\\
&y_{14} = y_{15} = (\frac{3}{2}d) - (a - \frac{d}{2}) = 2d - a
\end{aligned}$$

We must choose a to define our tiling, leaving d as our only free parameter, indicating that we are on a one-dimensional deformation space. □

Example 3.16. As before, we begin by putting a "reduced" dual graph on the tiling and labeling all squares, vectors, and loops. Note that we have three squares, nine vectors, and six loops, as expected.

We next define each of the vectors in the picture.

$$v_1 = \begin{pmatrix} a+b \\ y_1 \end{pmatrix} \quad v_2 = \begin{pmatrix} a+c \\ y_2 \end{pmatrix} \quad v_3 = \begin{pmatrix} a+b \\ y_3 \end{pmatrix}$$

$$v_4 = \begin{pmatrix} x_4 \\ 2a \end{pmatrix} \quad v_5 = \begin{pmatrix} a+c \\ y_5 \end{pmatrix} \quad v_6 = \begin{pmatrix} a+b \\ y_6 \end{pmatrix}$$

$$v_7 = \begin{pmatrix} a+c \\ y_7 \end{pmatrix} \quad v_8 = \begin{pmatrix} x_8 \\ b+c \end{pmatrix} \quad v_9 = \begin{pmatrix} x_9 \\ b+c \end{pmatrix}$$

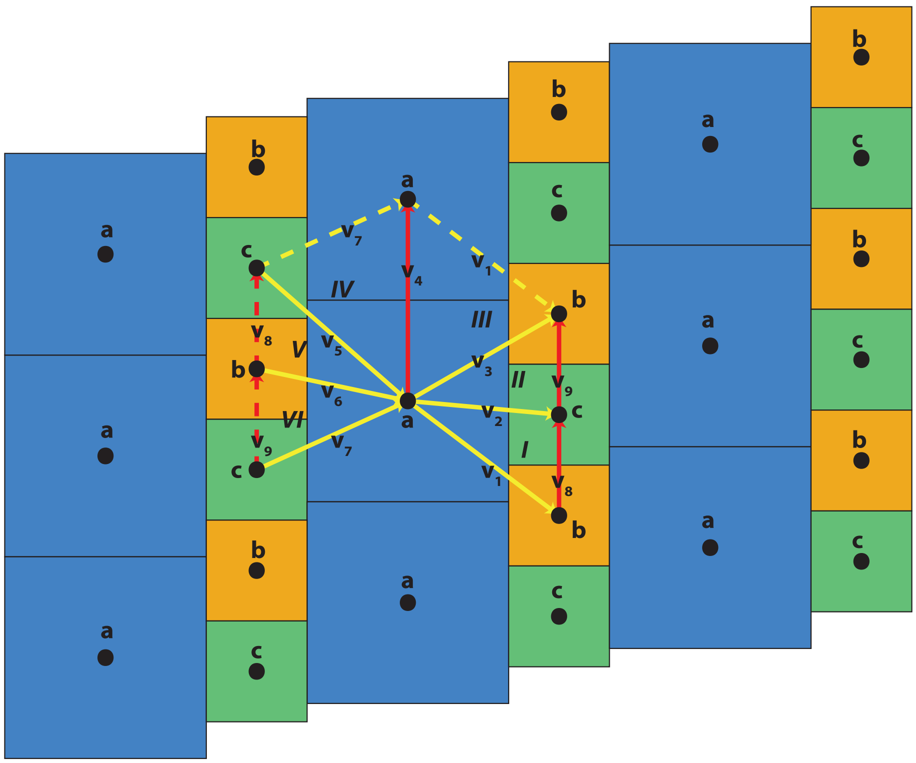

Figure 82: A tiling by three squares with all squares, vectors, and loops labeled

Now we write down the equation for each loop. Note there are six equations in this example.

$$
\begin{aligned}
&I : v_1 + v_8 - v_2 = 0 \\
&II : v_2 + v_9 - v_3 = 0 \\
&III : v_3 - v_1 - v_4 = 0 \\
&IV : v_4 - v_7 + v_5 = 0 \\
&V : v_6 - v_5 - v_8 = 0 \\
&VI : v_7 - v_6 - v_9 = 0
\end{aligned}
$$

Because of linear dependency, we will toss out one equation (say VI). Now we break up each vector equation into its x and y components, yielding ten equations.

$$
\begin{aligned}
&I_1 : a + b + x_8 - a - c = 0 \\
&I_2 : y_1 + b + c - y_2 = 0 \\
&II_1 : a + c + x_9 - a - b = 0 \\
&II_2 : y_2 + b + c - y_3 = 0 \\
&III_1 : a + b - a - b - x_4 = 0 \\
&III_2 : y_3 - y_1 - 2a = 0 \\
&IV_1 : x_4 - a - c + a + c = 0 \\
&IV_2 : 2a - y_7 + y_5 = 0 \\
&V_1 : a + b - a - c - x_8 = 0 \\
&V_2 : y_6 - y_5 - b - c = 0
\end{aligned}
$$

Next we reduce and rearrange each equation, writing the x_i or y_i terms on one side of the equality and the a, b, and c terms on the other side.

$$
\begin{aligned}
&I_1 \rightsquigarrow x_8 = c - b \\
&I_2 \rightsquigarrow y_2 - y_1 = b + c \\
&II_1 \rightsquigarrow x_9 = b - c \\
&II_2 \rightsquigarrow y_3 - y_2 = b + c \\
&III_1 \rightsquigarrow x_4 = 0 \\
&III_2 \rightsquigarrow y_3 - y_1 = 2a \\
&IV_1 \rightsquigarrow x_4 = 0 \\
&IV_2 \rightsquigarrow y_7 - y_5 = 2a \\
&V_1 \rightsquigarrow x_8 = b - c \\
&V_2 \rightsquigarrow y_6 - y_5 = b + c
\end{aligned}
$$

Right away note that I_1 and V_1 give:

$$x_8 = c - b = b - c \Rightarrow \{x_8 = 0 \text{ and } b = c\}.$$

Also note that $I_2 + II_2 = III_2$, which gives an important relation:

$$\begin{aligned}
&I_2 + II_2 = y_2 - y_1 + y_3 - y_2 = y_3 - y_1 = III_2 \\
\Leftrightarrow &I_2 + II_2 = (b+b) + (b+b) = 4b = 2a = III_2 \\
\Rightarrow &a = 2b
\end{aligned}$$

With these two relations, we can now write our ten equations in an even more reduced form:

$$\begin{aligned}
&I_1 \rightsquigarrow x_8 = 0 \\
&I_2 \rightsquigarrow y_2 - y_1 = 2b \Leftrightarrow y_2 - y_1 = a \\
&II_1 \rightsquigarrow x_9 = 0 \\
&II_2 \rightsquigarrow y_3 - y_2 = 2b \Leftrightarrow y_3 - y_2 = a \\
&III_1 \rightsquigarrow x_4 = 0 \\
&III_2 \rightsquigarrow y_3 - y_1 = 2a \\
&IV_1 \rightsquigarrow x_4 = 0 \\
&IV_2 \rightsquigarrow y_7 - y_5 = 2a \\
&V_1 \rightsquigarrow x_8 = 0 \\
&V_2 \rightsquigarrow y_6 - y_5 = 2b \Leftrightarrow y_6 - y_5 = a
\end{aligned}$$

We claim that a, y_2, and y_5 are free variables, and we will show this by explicitly writing each variable in terms of a, y_2, and y_5.

$$\begin{aligned}
a &= a \\
b &= \frac{a}{2} \\
c &= \frac{a}{2} \\
y_1 &= y_2 - a \\
y_2 &= y_2 \\
y_3 &= y_2 + a \\
x_4 &= 0 \\
y_5 &= y_5 \\
y_6 &= y_5 + a \\
y_7 &= y_5 + 2a \\
x_8 &= 0 \\
x_9 &= 0
\end{aligned}$$

We must choose a to define our tiling, so we are left with two free parameters, indicating that we are on a two-dimensional deformation space. $\square$

Notice how the deformation spaces from these examples all have a dimension of at least 1. Now let us shift gears and look at how the deformation spaces intersect and why these examples have such dimensions.

Usually (as one has seen from the moduli spaces above), a tiling doesn't intersect any additional deformation spaces other than the one on which it sits. This is the relatively "boring" case, but we can make it interesting by looking at what happen at each vertex of each square in the fundamental domain.

Notice that in a complete tiling, the area around each vertex must be covered. This means every part of the 360° degrees around each vertex must be covered. The vertex itself has a square coming from it, so this covers a fourth of the space, or 90°. Now we must find a way to cover the other 270°. Clearly we can have three more vertices all touch at this one point, each adding 90° to cover the space, so the area around the vertex will indeed be covered. Then at the vertex we have a "+" alignment. But we can also have an edge touch a vertex, and this will cover half of the space, or 180°. Then all we have left to cover is 90°, for which we can use another vertex. This creates a "T" alignment. We claim these are the only two alignments, and this follows directly from our progression above. The only two parts that can touch at a vertex are an edge and other vertices, so this means we can only add discrete values of 90° and 180° to get to 360°. Adding 180° to 180° does give 360°, but we are missing the important part of looking at what surrounds a vertex, so this case cannot happen. If we have an edge, the only way to get to 360° is to add on two more vertices, giving us the "T" alignment. If we do not have an edge, the only way to get to 360° is to add all vertices, giving us the "+" alignment.

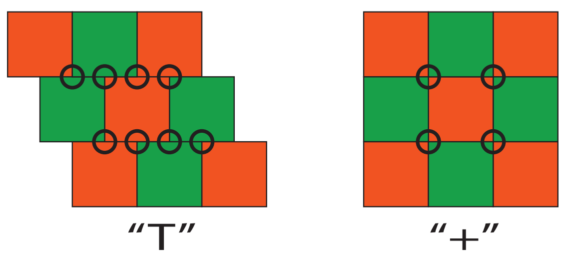

Figure 83: Examples of "T" alignments and "+" alignments

Proposition 3.17. *In a periodic tiling by squares, the only two possible alignments at each and every vertex are the "T" alignment and the "+" alignment.*

Proof. Our argument above suffices as a proof to this proposition; we showed how to get to the "T" and "+" alignments, and we showed that they were the only possible ones. □

So how does this relate to intersecting deformation spaces? Well, first we must look at only the alignments in and around the fundamental domain and disregard any redundant ones (the same thing we did when reducing the dual graph, its vectors, and its loops modulo translation). Since there are copies of the fundamental domain all around, we must make sure we don't count the same alignment twice: only once on one fundamental domain and not again in a different spot on a different fundamental domain. Drawing a piece of the tiling helps with this dilemma.

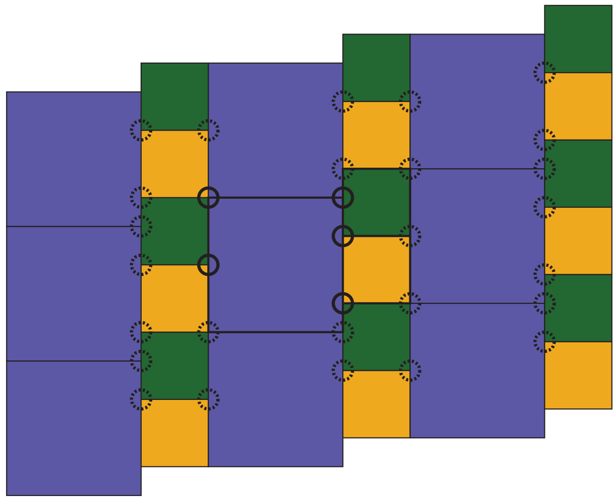

Figure 84: An example detailing how to accurately determine the number of alignments; note the dashed circles to indicate a repeated alignment (i.e. an alignment that is equivalent modulo translation to a previous one)

We claim that if our tiling only has “T” alignments, then we will stay on the same deformation space, yet if we “+” alignments, there exist different deformations to take us to different deformation spaces. If we have one “+” alignment, we have an intersection of two deformation spaces. If we have two “+” alignments, we have four different deformation spaces intersecting. With three “+”’s we have eight. There seems to exist a pattern with the amount of “+” alignments and the number of intersecting deformation spaces at that point. This does not hold true at limit points (obviously, since we are no longer dealing with the same number of squares). We have not proven this yet, but we will make a conjecture:

Remark 3.18. ***Deformation Space Intersection Conjecture (DSI Conjecture)***: For n number of “+” alignments in a given tiling ($n \geq 0$), there exist 2^n one-dimensional deformation spaces that intersect at that tiling.

We have an idea to start the proof, and it involves looking at the nearby moduli space. At a tiling with “+” alignments, the space nearby looks like the union of open subsets of finite dimensional vector spaces corresponding to the moduli spaces of tilings where each “+” alignment has been deformed into a “T” alignment. For n “+” alignments, there can be at most 2^n such spaces (since a single “+” alignment admits two “T” alignment deformations). The tricky part comes when we try to show this number is always reached. We can show this on a case-by-case basis, but we have not been able to prove this in general.

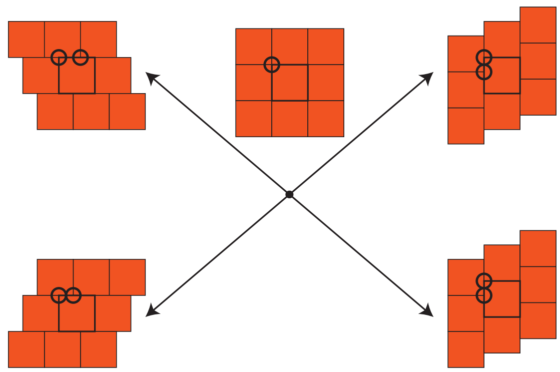

Figure 85: The intersecting moduli spaces in the one-square case; note there is one "+" alignment in the gridlock tiling and two one-dimensional moduli spaces that intersect at that tiling

We have to "maneuver" through the conjecture some to make it work for "+" alignments for tilings in a torus (and pay great attention to the one-dimensional detail). For instance, we can have the following case appear in the two-dimensional torus moduli space for two squares:

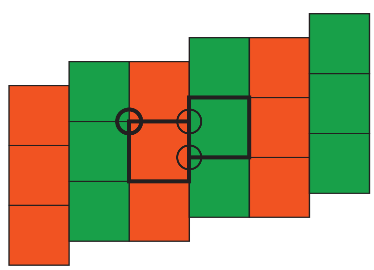

Figure 86: An example of a tiling located in the 2-D torus moduli space in the two-square tiling system; note there is one "+" alignment

In this case we have one "+" alignment, but we are already located deep inside a deformation space and are not near any others with which to intersect. How can this be? Well, recall that the two-dimensional torus is homeomorphic to $S^1 \times S^1$, which means we actually can think of this torus as two loops working together in this space, so anytime in this deformation space that we have a "+" alignment, we can think of the two S^1 loops crossing each other. This topological detail allows us to fit in the "+" alignments that we find while in toroidal moduli spaces. Note that in the three-square tilings V1, H1, V2, and H2, we have a three-dimensional torus as a moduli space, so we are working with three S^1

loops, which can cross each other in different ways, accounting for the various "+" alignments we may find.

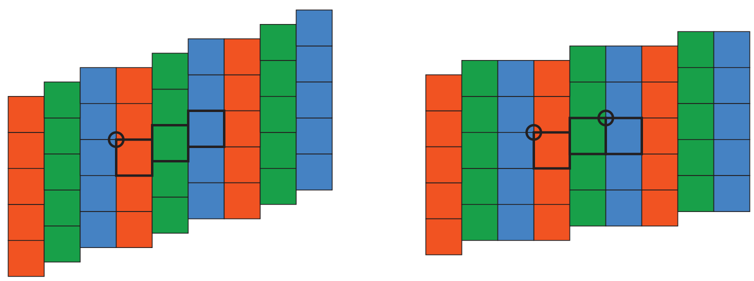

Figure 87: Examples of tilings in the 3-D torus attached to V1; note the tiling on the left has one "+" alignment and the tiling on the right has two "+" alignments

Another issue arises when we have the shared spaces that we brought up in the last section. Clearly, in the three-square tiling system, shared spaces are very prominent, with six shared spaces associated to every adjacent pair of the base tilings.

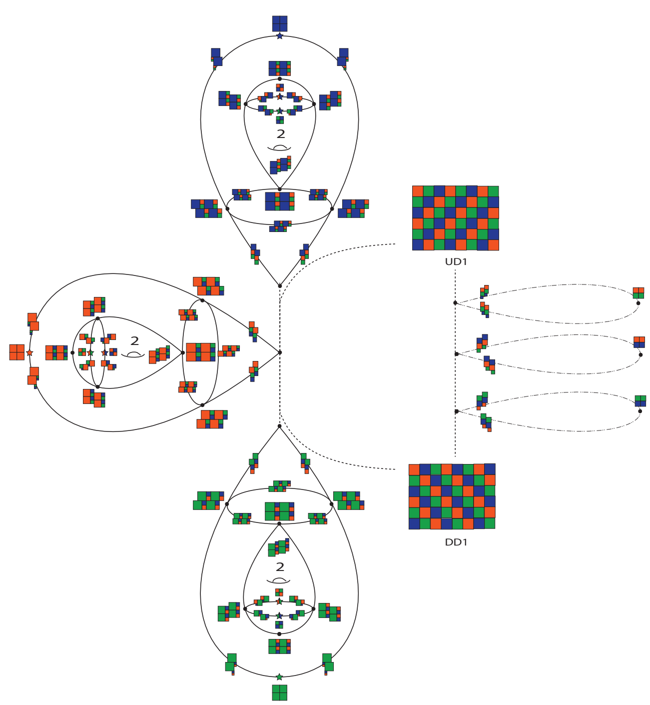

Figure 88: UD1 and DD1 and their six shared spaces

But if we count all the deformation spaces that surround a base tiling (let us take UD1 as an example), we get a massive number of fourteen deformation spaces: the horizontal-sliding space, the vertical-sliding space, the six shared spaces with DD1, and the six shared spaces with DD2.

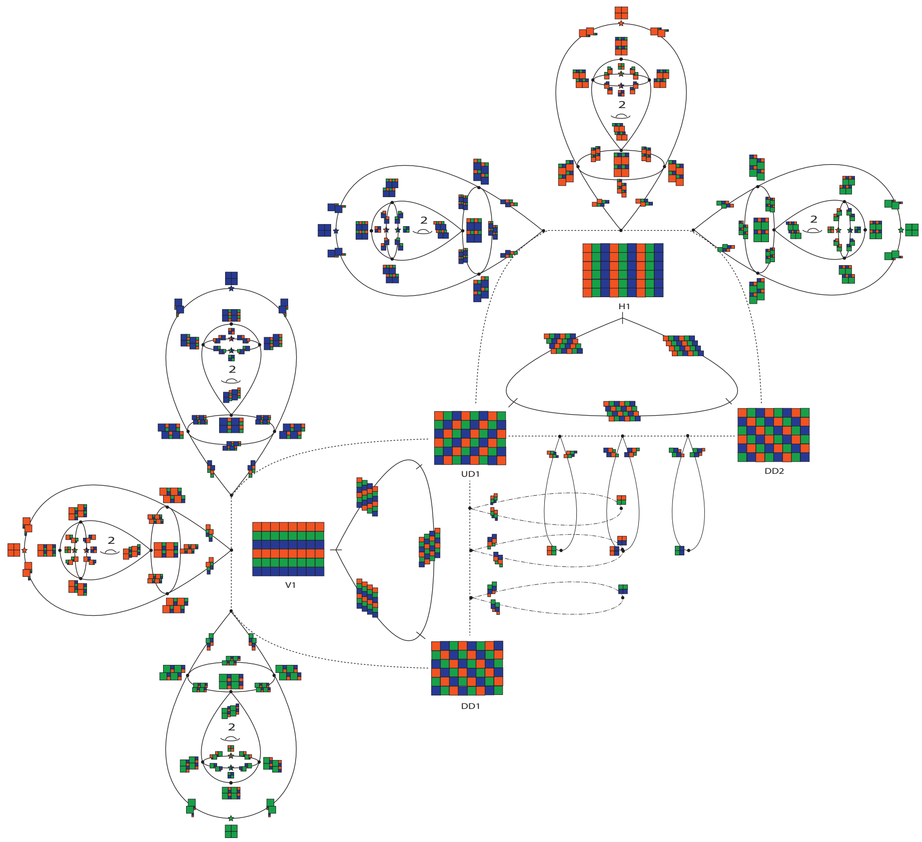

Figure 89: UD1 and all the deformation spaces that surround it

Since all four base cases have three "+" alignments, we expect $2^3 = 8$ intersecting deformation spaces. So what is the problem here? It comes from overcounting the shared spaces.

We should be counting each shared space once, but if we get fourteen for each of the four base cases, we have counted them all twice. For instance in the UD1 example, we included all the shared spaces with DD1. But if we look at DD1 and include all of the shared spaces with UD1, we have just become redundant. There are two ways around this. One is to include only one set of shared spaces. That is, for UD1 only including the shared spaces with DD2, for DD2 only including the shared spaces with UD2, for UD2 only including the shared spaces with DD1, and for DD1 only including the shared spaces with UD1. This way each shared space in counted only once, and each of the four base cases have eight intersecting deformation spaces.

This procedure is alright, but we would like a better way that incorporates both sets of shared spaces without overcounting. For this, we count the number of shared spaces between two tilings, divide by two, and add that number to the number of intersecting deformation spaces. It is the same idea in principal as the first, but in a more mathematical and property-preserving manner. That means for UD1, the shared spaces with DD1 account for $6 \div 2 = 3$ of the intersecting deformation spaces, and the shared spaces with DD2 account for $6 \div 2 = 3$ of the intersecting spaces as well. Adding this onto the two sliding deformation spaces, we get eight total intersecting spaces. This works for all four of the base cases, and the problem is settled.

One may ask what the difference is between overcounting the shared spaces and overcounting other spaces. Shouldn't it be that if two tilings are on the same deformation space that we add $\frac{1}{2}$ to each of their values for intersecting spaces? Clearly not, but it may not be obvious at first glance. In the case of shared spaces, we have two different tilings that act as the same point on a deformation space. This is why we divide by two in that case. For any other case like the UD1-DD1-V1 loop, UD1 and DD1 are on the same deformation space, but they are different points. This is why we do not divide by two (as we had been doing up to this point).

One final issue comes up with the intersection of a torus with another moduli space, namely a loop. Let us look at the intersection point of the 2-D torus with vertical band alignments for the two-square case, and then the intersection point of the 3-D torus with the vertical band alignments (the tiling V1) for the three-square case.

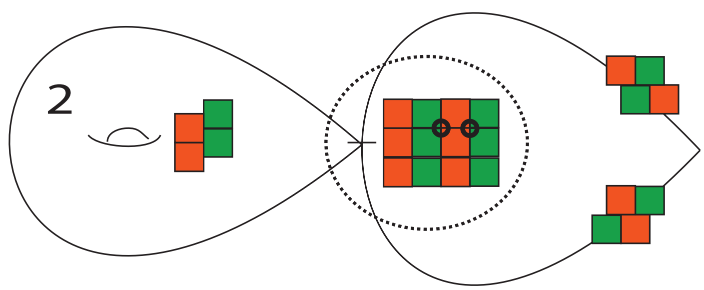

Figure 90: The 2-D torus, its neighboring loop, and the gridlock tiling at their intersection; note the gridlock tiling has two "+" alignments

Right away note that there are two "+" alignments, but we only have two intersecting moduli spaces, and our conjecture predicts there should be four. The issue arises with the underlying S^1 loops in the torus again. Instead of looking at the loop and the 2-D torus intersecting at a point, we must look at the loop intersecting with two other loops at a point, essentially quantizing the intersection of the loop and the 2-D torus. Yet these other two loops form the torus, so they behave differently. Let us call the torus loops α and β, and call the other moduli space loop M_0. Now we need four deformation spaces to intersect at this point. Clearly, we will have M_0 intersect. The trick comes when looking at the α- and β-loops. We will have the α-loop by itself intersect, and we will have the β-loop by itself intersect. We will also have a combination of the loops intersect: an α-β-loop so to speak. We include this combination to account for the direct product in $S^1 \times S^1$. Though we cannot realize these additional intersections when simply looking at the 2-D torus and the loop intersecting at a point, we can quantize the torus into its loops and see that mathematically, there are indeed four one-dimensional deformation spaces that intersect this point.

Now we can look at the case with three squares and the tiling V1:

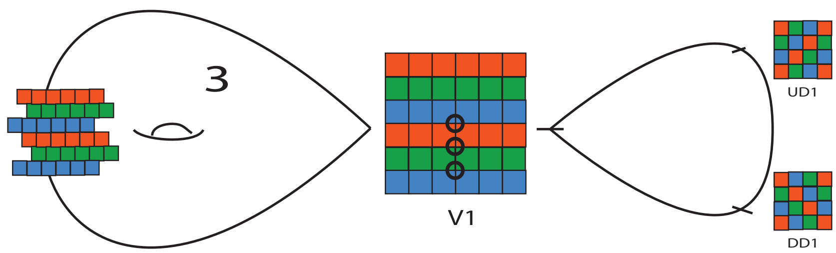

Figure 91: The 3-D torus, its neighboring loop, and the gridlock tiling (V1) at their intersection; note V1 has three "+" alignments

In this case, we have three "+" alignments, but again we only see two intersecting moduli spaces. This time our conjecture predicts there should be eight. Now, as before, we must break up the 3-D torus and look at its three underlying S^1 loops. Let us call them α, β, and γ, and let us call the other moduli loop M_0. Clearly we will have M_0 intersect, and clearly we will have the single α-loop, the single β-loop, and the single γ-loop intersect at this point as well. Now we must account for the direct product in $S^1 \times S^1 \times S^1$. With this, we find we also have the α-β-loop intersect, the α-γ-loop intersect, the β-γ-loop intersect, and the α-β-γ-loop intersect, all at this point V1. That gives us the eight intersecting one-dimensional deformation spaces we need, and our conjecture still holds.

The single internal loop inside of the 2-D tori in the three-square tiling system follows this precedence as well. Either side of internal loop accounts for one of the four possible intersecting moduli spaces, while the other three come from the combinations of loops in the 2-D torus.

We can also do this with a tiling by four squares that has vertical bands (similar to the V1 case). In this case we have a 4-D torus and a neighboring loop as our moduli spaces, and our tiling has four "+" alignments, which means we need sixteen intersecting one-dimensional deformation spaces.

We can label each of the four toroidal loops as before (and the original M_0 one as well) and look at all their different combinations as before. At the intersection, we can have the four loops by themselves, six different combinations with two loops, four different combinations with three loops, and one combina-

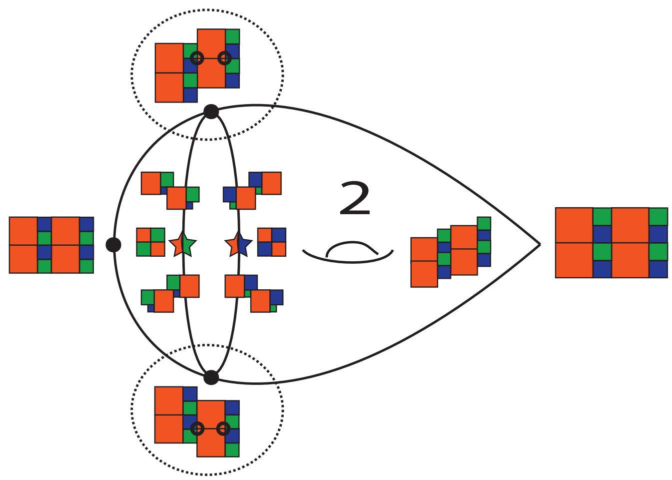

Figure 92: The internal loop inside of the 2-D torus in the three-square tiling system; note that the tiling at either intersection of the loop and the torus has two "+" alignments

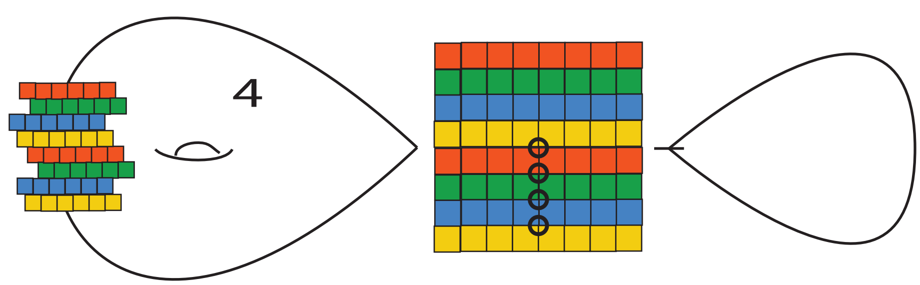

Figure 93: The 4-D torus, its neighboring loop, and the gridlock tiling at their intersection; note the gridlock tiling has four "+" alignments

tion with all four loops. Those fifteen intersecting loops plus the original M_0 loop account for all sixteen one-dimensional deformation spaces that intersect at that point.

Note that for this final issue of tori intersecting loops, the 2^n follows from the combinatorial formula involving the binomial coefficient:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

where k is the number of loops in consideration and n in the number of "+" alignments in the tiling (also the dimension of the torus in consideration). For any torus, $k = 0$ corresponds to M_0 intersecting, $k = 1$ corresponds to all single loops intersecting, $k = 2$ corresponds to all combinations of two loops, $k = 3$ corresponds to all combinations of three loops, and so on. It becomes clear that our conjecture will hold for all cases.

3.4 Conclusions

With the DSI Conjecture and the dual graph analysis, we can theoretically determine all parts of a tiling system for a finite number of squares. As we saw in the moduli space section, we were able to find the entire tiling systems for two and three squares without the use of the DSI Conjecture and the dual graph analysis, but when we think about four or five squares, the tiling system become so complicated that it would be extremely difficult to manually complete them. There are also so many more available tilings in four and five square cases (for instance, instead of just the "L"-shaped tiles and the rectangle tiles, "Z"-shaped tiles begin to appear) that it would be hard enough to merely think of all the different possible cases, let alone classify them and see how they interact with other deformation spaces.

With these two tools we described in the last section, we can look at a given tiling, use the dual graph analysis to determine the dimension of its deformation space and how to deform the tiling, and then use the DSI Conjecture to see how many other deformation spaces intersect the point we are looking at. Usually, we will have no "+" alignments and will have no intersections with other deformation spaces. But there will be tilings with these interesting "+" alignments after some deforming, and we will find beautiful mathematical patterns and topological objects through the various intersections of these deformation spaces as we further explore tilings by squares.

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